Second Memoir

Research on the
Primitive Parallelohedron

Georges Fedosevich Voronoi

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To God

Preface

Georges Fedosevich Voronoi † was born in Zhuravka, Poltava guberniya, Russia (now Ukraine) on 28th April, 1868. His father was a superintendent of Gymnasiums (senior secondary schools) in Kishinëv, a town in Moldova now, and in other towns in the then southern Ukraine.

He entered the Gymnasium in Priluki in 1885. After that he entered the Faculty of Physics and Mathematics at the University of St. Petersburg. He graduated from the department in 1885. Then he went on to do a master degree and completed it in 1894 His dissertation for the master degree was on algebraic integers associated with the roots of an irreducible cubic equation, that is a third-degree equation with a general form $ax^3 + bx^2 + cx + d = 0$. He became a professor in the Department of Pure Mathematics at University of Warsaw.

His doctoral dissertation submitted to the University of St. Petersburg in 1897 was on a generalisation of the algorithm of continued fractions, that is fractions written in the form $r = \frac{1}{a+\frac{1}{b+\frac{1}{c+\dots}}}$ and which terminates when r is rational. In the work for his Ph.D. he provided an algorithm ofr calculating fundamental units of a general cubic field for both a positive and negative discriminant. Both of his dissertations won the Bunyakovsky prize ‡ by the St. Petersburg Academy of Science.

The topic of these three papers of Voronoi goes back to Gauss (Johann Carl Friedrich Gauss, 1777–1855) and Hermite.

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† Sometimes written as Georgy Fedoseevich Voronoy. In French publications his last name is written *Voronoï*.

‡ Named after Viktor Yakovlevich Bunyakovskii (1804–1889).

New applications of continuous parameters to the theory of quadratic forms

Second Memoir

Research on the primitive parallelohedron

by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*]
[V. 134, 1908]

[translated by K N Tiyapan]

The well known method of reduction for the binary, ternary and quaternary positive quadratic forms † rests upon a property of the positive quadratic form, to know:

Every positive quadratic form $\sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ *has* n *variables in the set* E *composing all of the systems* $(x_1, x_2, \ldots, x_n)$ *of integers of the variables* x_1, x_2, $\ldots$, x_n n *consecutive minima*

$$M_1 \leq M_2 \leq \cdots \leq M_n$$

determined at condition which the determinant ω *of a system*

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1n}, l_{2n}, \ldots, l_{nn}) \quad (1)$$

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† *Lagrange, Recherches d'Arithmétique* [Studies in arithmetic] (*Oeuvres,* V. III, p. 695)

Gauß, Disquisitiones arithméticae (*Oeuvres,* V. I, art. 171, p. 146)

Lejeune-Dirichlet, Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen [On the reduction of the positive quadratic forms with three indeterminate integers] (*Oeuvres,* V. II, p. 41)

Minkowski, Sur la réduction des formes quadratiques positives quaternaires [On the reduction of the quaternary positive quadratic forms] (*Comptes Rendus des séances de l'Académie de Paris,* V. 96, p. 1205) *which represent these minima in the set E does not vanish.*

In all the cases where one has

$$\omega = \pm 1$$

one can transform the quadratic form $\sum\sum a_{ij} x_i x_j$ into an equivalent form by using a substitution

$$x_i = \sum_{k=1}^{n} l_{ik} x'_k \qquad (i = 1, 2, \ldots, n)$$

In the transformed form $\sum\sum a'_{ij} x_i x_j$, one will have

$$a'_{kk} = M_k \qquad (k = 1, 2, \ldots, n)$$

The form $\sum\sum a'_{ij} x_i x_j$ obtained is said to be reduced with respect to the consecutive minima.

The binary, ternary, and quaternary positive quadratic forms can be reduced with respect to the consecutive minima. ‡ The algorithm which one uses in doing the reduction of these forms is founded on the following theorem.

For a positive quadratic form

$$f(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j \quad (n = 2, 3, 4)$$

to be reduced with respect to the consecutive minima, it is necessary and sufficient that one has the inequalities

$$f(x_1, x_2, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) \geq a_{kk} \quad (k = 1, 2, \ldots, n) \tag{2}$$

and

$$a_{11} \leq a_{22} \leq \ldots \leq a_{nn} \tag{3}$$

---

‡ *Korkine* and *Zolotareff,* Sur les formes quadratiques positives. (*Mathematische Annalen,* V. 6, p. 336 and V. 11, p. 242)

which is valid for integers of the variables

$$x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \quad (k = 1, 2, \dots, n)$$

By letting

$$x_i = x'_i + \delta_i x'_k \quad \text{where } \delta_k = 0 \text{ and } i = 1, 2, \dots, n \tag{4}$$

one will determine for the given form $f(x_1, x_2, \dots, x_n)$ integers $\delta_1, \dots, \delta_{k-1}, \delta_{k+1}, \dots, \delta_n$ the condition of which the corresponding value $f(\delta_1, \dots, \delta_{k-1}, 1, \delta_{k+1}, \dots, \delta_n)$ would be smallest. By making successively $k = 1, 2, \dots, n$ and repeating the procedure stated, one will always transform the given form with the aid of the substitution (4) into a form which is no different from the reduced form except by a permutation of the coefficients $(n = 2, 3, 4)$

The procedure stated in the general case can not be carried on indefinitely and one will always arrive at an equivalent quadratic form $\sum a'_{ij} x_i x_j$ which verifies the inequalities (2) and (3), but one does not know from the number of variables $n > 4$ whether the coefficients a'_{kk} $(k = 1, 2, \dots, n)$ in the form obtained exhibit a system of consecutive minima, besides: one also does not know whether the reduction of every positive quadratic form with respect to the consecutive minima is possible.

One rids oneself of the described difficulty by changing the notation of system with n consecutive minima into nothing more than considering the systems (1) - which verify the equation

$$\omega = \pm 1 \, .$$

This is the method known as Hermite method † which has recently been improved by Mr. Minkowski in the memoir titled *Diskontinuitätsbereich für arithmetische Aquivalenz.* [Discontinuity domain for arithmetical equivalence] ‡ in the set E, the quadratique $\sum\sum a_{ij} x_i x_j$ being positive and $\alpha_1, \alpha_2, , \alpha_n$ any arbitrary parameters.

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† *Hermite, Extraits de lettres a Jacobi sur différents objet de la théorie des nombres* (This Journal, V. 40, p. 302)

‡ This Journal, V. 129, p.220

In the case $n = 2$, the problem put forward has been solved by Lejeune-Dirichlet and by Hermite §

By reflecting upon the principles which have served as basis in these researches of these two illustrious geometers, I have observed that the problem introduced is intimately connected to the problem of the reduction of positive quadratic form.

In effect, Lejeune-Dirichlet and Hermite have demonstrated the following theorem.

The conditions necessary and sufficient for which the inequality

$$ax^2 + 2bxy + cy^2 + 2\alpha y + 2\beta y \geq 0$$

holds, for any integer values of x and y, in general come down to six inequalities

$$\begin{cases} al^2 + 2blm + cm^2 \pm 2(\alpha l + \beta m) \geq 0, \\ al'^2 + 2bl'm' + cm'^2 \pm 2(\alpha l' + \beta m') \geq 0, \\ al''^2 + 2bl''m'' + cm''^2 \pm 2(\alpha l'' + \beta m'') \geq 0, \end{cases} \tag{5}$$

where the systems of integers

$$(l, m), (l', m') \text{ and } (l'', m'')$$

depend only on coefficients of the quadratic form (a, b, c).

By considering the parameters α and β as the Cartesian coordinates of a point (α, β) of the plane, one will determine by the inequalities (5) a hexagonal P which is formed by three pairs of parallel edges. The study of properties of the hexagon P plays an important role in the study of Lejeune-Dirichlet which has indicated two fundamental properties of the hexagon P.

I. There exists a group of translations of the hexagon P with the aid of which all the plane will be covered by the congruent hexagons.

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§ *Lejeune-Dirichlet*, Mémoire cited

Hermite, Sur la théorie des formes quadratique ternaires [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 178)

II. Any binary positive quadratic form can be transformed by an equivalent form (a, b, c) *satisfying the conditions*

$$a - b \geq 0, b \geq 0, c - b \geq 0. \qquad (6)$$

The hexagon P *corresponding to the form* (a, b, c)*, in the case*

$$a - b > 0, b > 0, c - b > 0,$$

is characterised by the systems

$$(1, 0), (0, 1), (1, -1) \qquad (7)$$

In the case $a - b = 0$*, or* $(b = 0)$*, or* $c - b = 0$*, the hexagon* P *reduces itself into a parallelogram.*

The inequalities (6) define a domain D of binary quadratic forms which is perfectly determined by the systems (7).

With the help of the substitution

$$x = x', \; y = -y',$$

one will transform the domain D by a domain D' defined by the inequalities

$$a + b \geq 0, -b \geq 0, c + b \geq 0 \qquad (8)$$

which is characterised by the systems

$$(1, 0), (0, 1), (1, 1)$$

One calls reduced by Selling['s method] the binary positive quadratic forms which verify the inequalities (8).‡

By effecting all the transformations of the domain D with the help of substitutions

$$x = px' + qy', y = p'x' + q'y'$$

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‡ *Selling,* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p.143)

of integer coefficients and of determinant ±1, one obtains a set (D) of domains of binary quadratic forms.

The set (D) of domains uniformly partitions the set of all the binary positive quadratic forms, that is to say: a form which is interior to any one domain D of the set (D) does not belong to any other domain of this set; a form which is interior to a face of the domain D belongs to only one other domain of the set (D) which is contiguous to the domain (D) by this face.

The results summarised have brought me to a new point of view on the problem of reduction of positive quadratic forms.

The problem of reduction of positive quadratic forms consist of a uniform partition of the set of positive quadratic forms with the help of domains of forms, determined using linear inequalities and enjoying the property that any substitution of integer coefficients and of determinant ± 1 *does not change the set* (D) *of these domains. By partitioning the set* (D) *into classes of equivalent domains and by choosing the representatives of all the classes*

$$D, D_1, \ldots, D_{m-1}, \tag{9}$$

one will call reduced the quadratic forms which belong to these domains.

One could attach the supplementary condition to the domains (9) by demanding: 1). that $m = 1, 2$, 2). that the positive quadratic forms interior to the domain D are not equivalent and lastly, 3). that the number of linear inequalities which define the domain D be the smallest one possible.

I hope to return another time to the problem posed of the reduction of positive quadratic forms.

In this mémoire, I restrict myself to the study of domains of quadratic forms which one obtains by generalising the results shown in studies of Lejeune-Dirichlet and of Hermite for the positive quadratic forms in any number of variables.

The hexagon of Lejeune-Dirichlet can be replaced for the positive quadratic forms of n variables by a convex polyhedron of the analytical space in n dimensions.

For a positive quadratic form $\sum\sum a_{ij}x_i x_j$, the corresponding polyhedron R presents a set of points (α_i) verifying the inequality

$$\sum\sum a_{ij}x_i x_j + 2\sum \alpha_i x_i \geq 0 \tag{10}$$

in the set E. The polyhedron R can be determined with the help of independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0,\ (k = 1, 2, \ldots, \tau)$$

the number 2τ of which does not exceed a limit

$$2\tau \leq 2(2^n - 1).$$

The systems of integers

$$\pm(l_{11}, l_{21}, \ldots, l_{n1}), \pm(l_{12}, l_{22}, \ldots, l_{n2}), \ldots, \pm(l_{1\tau}, l_{2\tau}, \ldots, l_{n\tau}) \tag{11}$$

defined by the corresponding equations

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} = 0$$

2τ faces in $n-1$ dimensions of the polyhedron R. As these faces partition themselves into τ pairs of parallel faces, I call parallelohedron the polyhedron R corresponding to any positive quadratic form.

The systems (11) enjoy many important properties.

1. For a system $(l_1, l_2, \ldots, l_n)$ *to belong to the series (11), it is necessary and sufficient that two systems* $(l_1, l_2, \ldots, l_n)$ *and* $(-l_1, -l_2, \ldots, -l_n)$ *are the only representations of the minimum of the form* $\sum\sum a_{ij}x_i x_j$ *in the set composed of all the systems of integers which are congruent to the system* $(l_1, l_2, \ldots, l_n)$ *by relation to the modulus 2, the system* $l_1 = 0, l_2 = 0, \ldots, l_n = 0$ *being excluded.*

2. Among the systems (11) are found all the representations of the arithmetical minimum of the positive quadratic form $\sum\sum a_{ij}x_i x_j$.

3. among the systems (11) are found all the systems (1) which represent n consecutive minima of the form $a_{ij}x_ix_j$.

4. All the determinants which one can form of any n systems belonging to the series (11) do not exceed in numerical value a limit $n!$.

By designating by the symbol S_ν the number of faces in ν dimensions $(\nu = 0, 1, 2, \ldots, n-1)$ of a parallelohedron R, I have found that

$$S_\nu \leq (n+1-\nu)\Delta^{(n=\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

By making $\nu = 0$ in this inequality, one obtains

$$S_0 \leq (n+1)!\,,$$

therefore the number of vertices of a parallelohedron R does not exceed a limit $(n+1)!$. By making $\nu = n-1$, one obtains

$$S_{n-1} \leq 2(2^n - 1)\,.$$

I demonstrate in this memoir that there exist parallelohedra, the symbol S_ν for which are expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

All these parallelohedra are primitive.

The notation of positive parallelohedra plays an important role in my studies.

I have arrived at the notation of primitive parallelohedra by observing that the parallelohedra possess Property I of hexagons of Lejeune-Dirichlet, in knowing:

I. There exists a group of transformations of a parallelohedron R with the help of which one uniformly fills the analytical space in n dimensions by the congruent parallelohedra.

Designate by (R) the set of parallelohedra which are defined by the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \,,$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers. Ay system (l_I) of integers characterise a parallelohedron of the set (R).

I demonstrate that the set (R) of parallelohedra corresponding to the various systems (l_i) of integers uniformly fills the space in n dimensions.

The corresponding group of translations of the parallelohedron R defined by the inequalities (10) is composed of vectors $[\lambda_i]$ which are determined by the equalities

$$\lambda_i = -\sum_{k=1}^{n} a_{ik}l_k, \quad (i = 1, 2, \ldots, n)$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

Any vertex (α_i) of parallelohedra of the set (R) belongs to at least $n+1$ parallelohedra. I call simple a vertex (α_i) which belongs only to $n+1$ parallelohedra of the set (R) and I establish a notion of primitive parallelohedron as follows:

One call primitive parallelohedron, a parallelohedron the vertices of which are simple.

All the parallelohedra which are not primitive are called nonprimitive. From this point of view, the hexagon of Lejeune-Dirichlet presents a primitive parallelohedron and each parallelogram is a nonprimitive parallelohedron in two dimensions.

Any nonprimitive parallelohedron is a boundary of primitive parallelohedra and can be considered as a case of degeneracy of primitive parallelohedra.

I divide the primitive parallelohedra into various types by characterising a type of primitive parallelohedra by a set (L) of simplexes correlative to the various vertices of parallelohedra which belong to the set (R).

An identical vertex (α_i) is determined by $n+1$ equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \quad (k = 0, 1, 2, \ldots, n)$$

In $n+1$ systems of integers

$$(l_{1k}, l_{2k}, \ldots, l_{nk}), \quad (k = 0, 1, 2, \ldots, n)$$

I make a simplex L correspond by defining it as a set of points which are determined by the equations

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik}, \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0.$$
$$(k = 0, 1, 2, \ldots, n, i = 1, 2, \ldots, n)$$

The set (L) of simplexes which are correlative to the vertices of the set (R) of primitive parallelohedra enjoys important properties.

1. The set (L) of simplexes uniformly partition the space of n dimensions.

2. By effecting the various translations of a simplex of the set (L) the length of vector $[l_i]$ which are determined by the arbitrary integers $l_1, l_2, \ldots, l_n$, one obtains a class of congruent simplexes which belong to the set (L).

3. The number of incongruent simplexes of the set (L) is finite.

Property II of hexagons of Lejeune-Dirichlet for the primitive parallelohedra can be generalised as follows:

II. All the quadratic forms which define the primitive parallelohedra belonging to the type characterised by the set (L) of simplexes are interior to a domain of quadratic form in $\frac{n(n+1)}{2}$ dimensions defined by linear inequalities.

I obtains the linear inequalities which define a domain D of quadratic forms corresponding to a set (L) of simplexes by examining the incongruent edges of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes.

An vertex (α_i) of primitive parallelohedra of the set (R) belongs to $n+1$ edges $[\alpha_i, \alpha_{ik}]$ of these parallelohedra $(k = 0, 1, 2, \ldots, n)$.

By putting

$$\alpha_{ik} - \alpha_i = p_{ik}\rho_k, \quad (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n)$$

one can determine the positive parameter ρ_k, of such manner that the numbers $p_{1k}, p_{2k}, \ldots, p_{nk}$ are integers and do not possesses common divisor. I demonstrate that the parameter ρ_k expressed by a linear function

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} \tag{12}$$

of coefficients of the given quadratic form $\sum\sum a_{ij}x_i x_j$, the coefficients

$$p_{ij}^{(k)} = p_{jk}^{(k)}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

being rational.

I call regulator of the edge $[\alpha_i, \alpha_{ik}]$, the function ρ_k determined by the formula (12); the system (p_{ik}) is called characteristic of the edge

$$[\alpha_i, \alpha_{ik}]. \quad (k = 0, 1, 2, \ldots, n)$$

As the edge $[\alpha_i, \alpha_{ik}]$ is correlative to a face P_k of $n-1$ dimensions of the simplex L which is correlative to the vertex (α_i), I call the function (12) *regulator of the face* P_k and the system $\pm(P_{ik})$ *characteristic of the face* P_k *of the simplex* L $(k = 0, 1, 2, \ldots, n)$

By designating by

$$\rho_k \text{ and } \pm(p_{ik}), \quad (k = 1, 2, \ldots, \sigma)$$

the regulators and the characteristics of all the incongruent faces in $n-1$ dimensions of the set (L) of simplexes, I demonstrate the following important theorem:

The domain of quadratic forms which is characterised by the set (L) *of simplexes is defined by the linear inequalities*

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

All the domains of quadratic forms which I have studied in this memoir possess a remarkable property: they are simple domains, that is to say the number of independent inequalities which define them is equal to $\frac{n(n+1)}{2}$.

Another coincidence has attracted my attention for a long time: that is the relation which exists within the results shown in this memoir and those which have been obtained in my first memoir titled: "On some properties of perfected positive quadratic forms" † I have observed that the set of characteristics $\pm(p_{ik}), k = 1, 2, \ldots, \sigma$ is nothing but the set of all the representations of the minimum of a perfect quadratic form φ. Thee domain D either coincides well with the domain R corresponding to the perfect form φ, or presents well a group of this domain.

Despite all my effort, I have not succeeded in discovering the tie which attaches the two problems shown and which seem to be so different, abstraction made of a remarkable formula

$$\sum\sum a_{ij} x_i x_j = \frac{1}{(n-1)!} \sum_{k=1}^{\sigma} \rho_k \omega_k (p_{1k} x_1 + p_{2k} x_2 + \cdots + p_{nk} x_n)^2$$

which supplies the expression of an arbitrary quadratic form $\sum\sum a_{ij} \cdot x_i x_j$ in function of the regulators $\rho_k\, (k = 1, 2, \ldots, \sigma)$ which are determined by the formula (12).

In this formula $\omega_k\, (k = 1, 2, \ldots, \sigma)$ are positive integers which depend only on corresponding faces of simplexes of the set (L).

To the various types of primitive parallelohedra corresponds a set (D) of domains of quadratic forms. The set (D) uniformly partitions the set of all the positive quadratic forms in n variables.

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† This Journal, V. 133, p. 97

I show in this mémoire an algorithm, by the aid of which one can determine all the domains of forms which are contiguous to a domain of the set (D) by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. This algorithm comes down to a certain reconstruction of the set (L) of simplexes by another set (L').

The set (D) of domains of forms transforms into itself by all the substitutions of integer coefficients and of determinant ±1. By dividing the set (D) into classes of equivalent domains, one obtains with the aid of the algorithm shown the representatives

$$D, D_1, \ldots, D_{m-1}$$

of various classes of domains belonging to the set (D).

By calling reduced the quadratic forms which belong to the domains obtained, one establishes a new method of reduction of positive quadratic forms.

I have applied the general theory shown to the study of two types of primitive parallelohedra of the space in n dimensions which correspond to the principal domain of quadratic forms and to the domains which are contiguous to the principal domain by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. The principal domain is defined by the inequalities

$$\begin{aligned} \sum_{k=1}^{n} a_{ik} \geq 0, &\quad (i = 1, 2, \ldots, n) \\ -a_{ij} \geq 0. &\quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j) \end{aligned}$$

I study in detail the parallelohedra of the space in 2, 3 and 4 dimensions.

In the space in 2 dimensions, there is only one type of primitive parallelohedra, provided that one does not consider as different the equivalent types; it is the hexagon of Lejeune-Dirichlet.

The set (D) of domains is composed in this case of a single class, the representative of which is the principal domain defined by the inequalities (8).

In the space in 2 dimensions, there is only one single space of primitive parallelohedra – it is the parallelogram.

In the space in 3 dimensions, there is only one single type of primitive parallelohedra – it is a polyhedron of 14 faces, 8 of which are hexagonal and 6 of which are parallelogrammatic.

The set (D) of domains is composed in this case of a single class, the representative of which is the principal domain. By calling reduced a ternary positive quadratic form $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ which belongs to the principal domain determined with the help of inequalities

$$a + b' + b'' \geq 0,$$

$$a' + b'' + b \geq 0,$$

$$a'' + b + b' \geq 0,$$

$$-b \geq 0, \quad -b' \geq 0, \quad -b'' \geq 0,$$

one will arrive at the method of reduction of ternary positive quadratic forms due to Selling. ‡

In the space in 3 dimensions, there are 4 spaces of primitive parallelohedra, they are :

1). the parallelepiped,

2). the prism of hexagonal base,

3). the parallelegrammatic dodecahedron and

4). the dodecahedron in 4 hexagonal faces and 8 parallelogrammatic faces.

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‡ *Selling,* Mémoire cited

In the space of 4 dimensions, there are three types of primitive parallelohedra. The set (D) of domains is composed of three classes of domains of quaternary quadratic forms.

I have determined the three representatives of these classes

$$D, D', D''.$$

By calling as reduced the quaternary positive quadratic form which belong to the domains D, D', D'', I have arrived at a modification of the methods of reduction of quaternary positive quadratic forms due to Mr. Charve. †

By virtue of this theorem, the problem of uniform partition of the space in n dimensions by congruent primitive parallelohedra always comes down to the study of parallelohedra corresponding to the positive quadratic forms.

I am inclined to think, without being able to demonstrate, that the theorem introduced is also true for the nonprimitive parallelohedra.

The parallelohedra of the space in 2 and in 3 dimensions have been studied by Mr. Fedorow ¶ which has

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† *Charve, De la réduction des formes quadratiques quaternaires positives* [Of the reduction of quaternary quadratic forms] (*Comptes-Rendus des séances de l'Academie du Paris*), V. 92, p.782 and *Annales de l'École Normale supérieure*, 2^{nd} *serie*, V. XI, p.119

¶ *Fedorow,* Basic principles in the theory of diagrams. St. Petersbourg, 1885 (in Russian)

Fedorow, Reguläre Plan- und Raumteilung. [Regular planar and space partition] (*Abhandlungen der K. bayer. Akademie der Wiss.* II Cl., XX Bd. II Abt. München, 1899)

See also: *Minkowski, Allgemeine Lehrsätze über die convexen Polyeder.* [General theorems on the onvex polyhedron] (*Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathem.-Physikalische* discovered with the help of purely geometrical considerations, the exitence of two spaces of parallelohedra in the space in 2 dimensions and the existence of five spaces of parallelohedra in the space in 3 dimensions. Mr. Fedorow has demonstrated that there is no other parallelohedra in the space of 2 and of 3 dimensions.

The parallelohedra in 3 dimensions of Mr. Fedorow play an important role in the theory of the structure of crystals. §

First part

Uniform partition of the analytical space in n dimensions with the aid of translations of the same convex polyhedron

Section I

General properties of parallelohedra

On the convex polyhedra in n dimensions

1

One will call point of the analytical space in n dimensions any systems $(x_1, x_2, \ldots, x_n)$, or simply (x_i), of real values of variables $x_1, x_2, \ldots, x_n$.

Consider a system of linear inequalities

$$a_{0k} + \sum_{i=1}^{n} a_{ik} x_i \geq 0 \quad (k = 1, 2, \ldots, \sigma) \tag{1}$$

Klasses, 1897, p.198)

§ See: *Fedorow, Courses in Crystallography.* St. Petersbourg, 1901 (in Russian)

Soret, Cristallographie physique. [Physical crystallography] Genève, 1894.

Schönflies, Kristallsysteme und Kristallstruktur. [Crystal systems and crystal structure] Leipzig, 1891

Sommerfeldt, Physikalische Kristallographie. [Physical crystallography] Leipzig, 1907.

of any real coefficients.

One will say that the set R of points verifying the inequalities (1) is of n dimensions, if there exist points satisfying the conditions

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call them point, interior to the set R.

Fundamental principle. † *For the set R of points verifying the inequalities (1) to be of n dimensions, it is necessary and sufficient that the equation*

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0$$

does not reduce into an identity so long as all the parameters $\rho_0, \rho_1, \ldots, \rho_\sigma$ are positive or zero.

Definition I. *One will call convex polyhedron any set of points verifying a system of linear inequalities, on condition that this set be bounded and of n dimensions.*

2

Let us suppose that the inequalities (1) define a convex polyhedron R and suppose that all the inequalities (1) be independent. In such case, the polyhedron R possesses σ faces in $n-1$ dimensions which are defined by the corresponding equations

$$a_{0k} + \sum a_{ik} x_i = 0. \quad (k = 1, 2, \ldots, \sigma)$$

Definition II. *Suppose that a point (α_i) belonging to R verifies the equations*

$$a_{0r} + \sum a_{ir} x_i = 0, \quad (r = 1, 2, \ldots, \mu) \qquad (2)$$

---

† *The principle announced differs only in the formulation from the fundamental principle explained in my first memoir titled: On some properties of perfect positive quadratic forms. (This journal, V. 133, p. 113)*

and that one had the inequalities

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = \mu + 1, \ldots, \sigma)$$

Designate by ν the number of dimensions of the set $P(\nu)$ composed of points belonging to R and verifying the equations (2). One will call face in ν dimensions of the polyhedron R *the set* $P(\nu)a, (\nu = 0, 1, 2, \ldots, n-1)$.

In the case $\nu = 1$, one will call edge of the polyhedron R a face $P(1)$ and in the case $\nu = 0$, one will call vertex of the polyhedron a face $P(0)$.

For more generality in the notations, one will designate by the symbol $P(n)$ the polyhedron R itself.

Under this restriction, one can introduce the following proposition:

Any point belonging to the polyhedron R is interior to a face $P(\nu)$ of that polyhedron, where $\nu = 0, 1, 2, \ldots, n$.

3

Let us suppose that the polyhedron R possesses s vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is}).$$

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{im})$$

all the vertices of R which verify the equations (2).

Theorem. † *The face $P(\nu)$ in ν dimensions ($\nu = 0, 1, 2, \ldots, n$) of the polyhedron R defined by the equations (2) presents a set of points determined by the aid of equalities*

$$x_i = \sum_{r=1}^{m} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r \geq 0.$$

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† *See my mémoire cited, Number 12*

$$(r = 1, 2, \ldots, m)$$

Set of domains in n dimensions corresponding to the different vertices of a convex polyhedron.

4

Let us suppose a vertex (α_i) of the polyhedron R be determined by the equations

$$a_{0k} + \sum a_{ik} x_i = 0. \quad (k = 1, 2, \ldots, \mu) \tag{1}$$

Definition. *One will call domain corresponding to the vertex (x_i) the set A of points determined with the help of equalities*

$$x_i = \sum_{k=1}^{\mu} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \mu) \tag{2}$$

Designate by

$$A_1, A_2, \ldots, A_s \tag{3}$$

the domains corresponding to the different vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

of the polyhedron R. By virtue of the definition established, the set (3) of domains enjoys the following properties:

I. All the domains of the set (3) are in n dimensions.

Let us suppose the domain A determined by the equalities (2) be not in n dimensions.

All the points (x_i) belonging to the domain A verify at least one linear equation

$$\sum p_i x_i = 0.$$

By virtue of (2), one will have

$$\sum p_i a_{ik} = 0. \ (k = 1, 2, \ldots, \mu) \tag{4}$$

As the equations (1) define a vertex (α_i) of the polyhedron R, one will find among the systems

$$(a_{11}, \dots, a_{n1}), (a_{12}, \dots, a_{n2}), \dots, (a_{1\mu}, \dots, a_{n\mu})$$

n systems the determinant of which is not zero; it follow that the equalities (4) are impossible.

II. Any point of the space in n dimensions belong to at least one domain of the set (3).

Let (α_i) be an arbitrary point. Examine the sum

$$\sum a_i \alpha_{ik}, \quad (k = 1, 2, \dots, s)$$

and suppose that the smallest sum $\sum a_i x_i$ correspond to the vertex (α_i) defined by the equations (1). One will have the inequalities

$$\sum a_i \alpha_{ik} \geq \sum a_i \alpha_i. \quad (k = 1, 2, \dots, s)$$

By virtue of the theorem of Number 3, one obtains

$$\sum a_i x_i \geq \sum a_i \alpha_i,$$

for any point (x_i) belonging to the polyhedron R.

One concludes that the inequalities

$$\sum a_i \alpha_i - \sum a_i x \geq 0 \text{ and } a_{0k} + \sum a_{ik} x_i \geq 0 \ (k = 1, 2, \dots, \sigma)$$

can not define a polyhedra in n dimensions and, by virtue of the fundamental principle of Number 1, one will have an identity

$$\rho_0 + \rho(\sum a_i \alpha_i - \sum a_i x_i) + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0,$$

where

$$\rho_0 \geq 0, \rho \geq 0, \rho_k \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

By making $x_i = \alpha_i$ in this identity, it will become

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} \alpha_i) = 0,$$

and as according to the supposition made

$$a_{0k} + \sum a_{ik}\alpha_i > 0$$

as long as $k = \mu + 1, \dots, \sigma$, it is necessary that

$$\rho_0 = 0, \rho_{\mu+1} = 0, \dots, \rho_\sigma = 0,$$

therefore

$$\rho(\sum a_i\alpha_i - \sum a_i x_i) + \sum_{k=1}^{\mu} \rho_k(a_{0k} + \sum a_{ik}x_i) = 0.$$

One draws

$$a_i = \sum_{k=1}^{\mu} \frac{\rho_k}{\rho} a_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0, \ (k = 1, 2, \dots, \mu)$$

therefore the points (α_i) belongs to the domain A.

III. A point which is interior to a face $A(\nu)$ $(\nu = 0, 1, 2, \dots, n)$ of any domain of the set (3) belongs only to the domains of the set (3) which are contiguous by the face $A(\nu)$.

Suppose the point (a_i) be interior to a face $A(\nu)$ of the domain A.

By designating with

$$(a_{i1}), (a_{i2}), \dots, (a_{i\tau}), \tau \leq \mu$$

the points which characterise the face $A(\nu)$, one can put †

$$a_i = \sum_{k=1}^{\tau} \rho_k a_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \dots, \tau)$$

Suppose that the point (a_i) be interior to another face $A'(\nu')$ of a domain A' which corresponds to a vertex (α_i'). One can put

$$a_i = \sum_{h=1}^{\tau'} \rho_h' a_{ih}' \text{ where } \rho_h' > 0. \ (h = 1, 2, \dots, \tau')$$

---

† See my mémoire cited, Number 13

By virtue of these equalities, one will have an identity

$$\rho_0 + \sum_{h=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} x_i) = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} x_i) \qquad (5)$$

By making within this identity $x_i = \alpha_i$, one obtains

$$\rho_0 = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} \alpha_i),$$

and it results that

$$\rho_0 \geq 0.$$

By making within the identity (5) $x_i = x'_i$, one obtains

$$\rho_0 + \sum_{k=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} \alpha'_i) = 0;$$

consequently $\rho_0 = 0$ and

$$a_{0k} + \sum a_{ik} \alpha'_i = 0. \quad (k = 1, 2, \ldots, \tau)$$

In the same manner, one finds

$$a'_{0h} + \sum a'_{ih} \alpha_i = 0. \quad (h = 1, 2, \ldots, \tau')$$

One concludes that the two faces $A(\nu)$ and $A'(\nu')$ coincide. ‡

By virtue of properties demonstrated of the set (3) of domains, one will say that this set uniformly *partitions* the space in n dimensions.

5 *Definition of the group of vectors*

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‡ See my mémoire cited, Number 20, p. 133

Definition I. *One will call vector the set of points determined with the help of equalities*

$$x_i = x_i + u(\alpha_i' - \alpha_i) \text{ where } 9 \leq u \leq 1, \tag{1}$$

(α_i) and (α_i') being any two different points.

One will designate the vector determined with the help of equalities (1) by the symbol $[\alpha_i, \alpha_i']$. In this case $\alpha_i = 0\,(i = 1, 2, \ldots, n)$, one will designate the corresponding vector by the symbol $[\alpha_i']$ and one will call it vector of the point (α_i').

Definition II. *Suppose that*

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}] \tag{2}$$

be the vectors of arbitrary points $(\lambda_{i1}), (\lambda_{i2}), \ldots, (\lambda_{im})$.- One will call group of vectors *the set G of vectors determined with the help of equalities*

$$\lambda_i = \sum_{k=1}^{m} l_k \lambda_{ik},$$

$l_1, l_2, \ldots, l_m$ being of arbitrary integers.

One will call *basis of the group G of vectors* the vectors (2).

Translation of polyhedra.

6

Definition. Effect a linear transformation of a polyhedron R with the help of a substitution

$$x_i = x_i' - \lambda_i, \ (i = 1, 2, \ldots, n) \tag{1}$$

the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ being arbitrary. One will say that one has effected a translation of the polyhedron R the length of the vector $[\lambda_i]$.

Suppose that the polyhedron R be determined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

The transformed polyhedron R' will be determined, by virtue of (1), by the inequalities

$$a_{0k} + \sum a_{ik}(x_i - \lambda_i) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call congruent the polyhedra R and R'.

7

Let G be a group of vectors. By effecting the different translations of the poly hedron R the length of vectors belonging to the group G, one will form a set R of congruent polyhedra.

One will say that the set (R) of congruent polyhedra uniformly partition the space in n dimensions in the following conditions.

I. Any point of the space in n dimensions belongs to at least one polyhedron of the set (R).

II. A point which is interior to any one face $P(\nu)$ $(\nu = 0, 1, 2, \ldots, n)$ *of a polyhedron of the set* (R) *belongs to only the polyhedrons of the set* (R) *which are contiguous by the face* $P(\nu)$.

Definition of parallelohedra

8

Definition. One will call parallelohedron any convex polyhedron R possessing a group G of translations with the aid of which one can uniformly fill the space in n dimensions by the polyhedra congruent to the polyhedron R.

By virtue of the definition established, the parallelohedra possess an important property, in knowing:

By effecting a linear transformation of a parallelohedron with the help of a substitution by any real coefficients

$$x_i = \alpha_{i0} + \sum_{k=1}^{n} \alpha_{ik} x'_k, \qquad (i = 1, 2, \ldots, n)$$

one obtains a convex polyhedron which is also a parallelohedron.

Observe that by virtue of the definition established, any parallelohedron of the space in n dimensions is a parallelohedron.

9 *Properties of the group of vectors of a parallelohedron.*

Suppose that a parallelohedron R be defined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by G the group of the parallelohedron R and suppose that the group G possesses the basis

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}]. \tag{2}$$

All the vectors which form the basis of the group G can not verify the same linear equation

$$\sum p_i \lambda_i = 0,$$

because otherwise the set (R) of congruent parallelohedra corresponding to the group G would not fill the space in n dimensions.

One concludes that among the vectors (2) there are n vectors

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}] \tag{3}$$

the determinant $\pm\Delta$ of which is not zero; one will call them independent.

Theorem I. *The numerical value* Δ *of the determinant of* n *independent vectors possesses a limit*

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n \,.$$

Let (α_i) be any point which is interior to the parallelohedron R. Introduce within our researches a parallelepiped K determined with the aid of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \quad (i = 1, 2, \ldots, n) \tag{4}$$

where

$$-\delta \leq u_k \leq \delta. \quad (k = 1, 2, \ldots, n) \tag{5}$$

One can choose the positive parameter δ in such manner that all the points of the parallelohedron R defined by the inequalities (1) belong to the parallelepiped K.

Take a positive integer m and determine $(m+1)^n$ systems $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequalities

$$0 \leq l_k \leq m. \quad (k = 1, 2, \ldots, n) \tag{6}$$

Designate by

$$\lambda_i^{(h)} = \sum_{k=1}^{n} l_k^{(h)} \lambda_{ik}, \quad (h = 1, 2, \ldots, (m+1)^n) \tag{7}$$

$(m+1)^n$ corresponding vectors belonging to the group G.

By applying the translations of the parallelohedra R the length of vectors (7), one obtains $(m+1)^n$ different parallelohedra of the set (R):

$$R^{(h)}. \quad (h = 1, 2, \ldots, (m+1)^n) \tag{8}$$

Designate by H a parallelepiped which is determined by the equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \ (i = 1, 2, \ldots, n) \tag{9}$$

where

$$-\delta \leq u_k \leq m + \delta. \quad (k = 1, 2, \ldots, n) \tag{10}$$

I argue that all the points of parallelohedron (8) belong to the parallelepiped H. In effect, let $(x_i^{(h)})$ be any point of the parallelohedron $R^{(h)}$ $(h = 1, 2, \ldots, (m+1)^n)$. By posing

$$x_i = x_i^{(h)} - \lambda_i^{(h)}, \quad (i = 1, 2, \ldots, n) \tag{11}$$

one obtains a point (x_i) belonging to the parallelohedron R which is congruent to the point given $(x_i^{(h)})$. By virtue of (4), (7) and (11), one obtains

$$x_i^{(h)} = \alpha_i + \sum_{k=1}^{n} (l_k + u_k)\lambda_{ik},$$

and by (5) and (6), it becomes

$$-\delta \leq l_k + u_k \leq m + \delta, \quad (k = 1, 2, \ldots, n)$$

thus the point $(x_i^{(h)})$ belongs to the parallelepiped H.

It follows that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n \geq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n. \quad (h = 1, 2, \ldots, (m+1)^n)$$

By observing that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n = \Delta(m + 2\delta)^n$$

and that

$$\int_{(R^h)} dx_1 dx_2 \cdots dx_n = \int_{(R)} dx_1 dx_2 \cdots dx_n, \quad (h = 1, 2, \ldots, (m+1)^n)$$

one obtains

$$\Delta(m + 2\delta)^n \geq (m+1)^n \int_{(R)} dx_1 dx_2 \cdots dx_n.$$

By making the number m increase indefinitely, one finds

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n.$$

10

Theorem II. *The group G of vectors of a parallelohedron possesses basis formed by n independent vectors.*

Designate by G' a group of vectors having the basis (3). It can be that the two groups G and G' coincide. In that case n vectors (3) present a basis of the group G.

By supposing the contrary, one will have among the vectors (2) at least one vector $[\lambda'_i]$ which does not belong to the group G'. By putting

$$\lambda'_i = \sum_{k=1}^{n} l'_k \lambda_{ik} ,$$

one will have among the numbers $l'_1, l'_2, \ldots, l'_n$ at least one number which is fractional.

Designate by $l_1, l_2, \ldots, l_n$ the integers verifying the inequalities

$$|l'_k - l_k| \leq \frac{1}{2} \quad (k = 1, 2, \ldots, n)$$

and suppose that $l'_r - l_r \neq 0$.

By designating

$$\lambda'_{ik} = \lambda_{ik} \, , \; (k = 1, 2, \ldots, n, k \neq r) \text{ and } \lambda'_{ir} = \lambda'_i - \sum l_k \lambda_{ik} , ,$$

one obtains a system of n independent vectors

$$[\lambda'_{i1}], [\lambda'_{i2}], \ldots, [\lambda'_{in}]$$

belonging to the group G, the determinant $\pm\Delta'$ of which verifies the inequality

$$0 < \Delta' \leq \frac{1}{2}\Delta .$$

The procedure explained can not be prolonged indefinitely, by virtue of Theorem I, therefore one will always obtain a system of n vectors forming the basis of the group G.

11

Theorem III. *The numerical value Δ of the determinant of a system of n vectors forming the basis of the group G is expressed by the formula*

$$\Delta = \int_{(R)} dx_1 \cdot dx_2 \cdots dx_n \, .$$

suppose that the system (3) of n vectors presents a basis of the group G.

Introduce in okur studies a parallelepiped H' determined with the help of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \; (i = 1, 2, \ldots, n) \qquad (12)$$

where

$$\delta \leq u_k \leq m - \delta \, . \, (k = 1, 2, \ldots, n) \qquad (13)$$

I argue that any point of the parallelepiped H' belongs to at least one of parallelohedron (8). In effect, let (x_i') be any point of the parallelepiped H'.

Designate by R^0 a parallelohedron of the set (R) to which belongs the point (x_i'). Let $[\lambda_i]$ be the vector which defines a translation of the parallelohedron R to R^0. By putting

$$x_i = x_i' - \lambda_i \, , \qquad (14)$$

one obtains a point (x_i) belonging to the parallelohedron R which is congruent to the point x_i'. By virtue of the supposition made, the vector $[\lambda_i]$ can be determined by the equalities

$$\lambda_i = \sum_{k=1}^{n} l_k \lambda_{ik} \, . \qquad (15)$$

As the point (x_i') belongs to the parallelepiped H', one will present the equalities (14), by (12) and (15), in the following form:

$$x + i = \alpha_i + \sum_{k=1}^{n} (u_k - l_k) \lambda_{ik} \, .$$

The point (x_i) belonging to the parallelohedron R belongs also, by virtue of the supposition made, to the parallelepiped K determined by the equalities (4), by condition of (5). It follows that

$$-\delta \leq u_k - l_k \leq \delta\,, \quad (k = 1, 2, \ldots, n)$$

and as, by (13),

$$\delta \leq u_k \leq m - \delta\,, \quad (k = 1, 2, \ldots, n)$$

it becomes

$$0 \leq l_k \leq m, \quad (k = 1, 2, \ldots, n)$$

therefore the vector $[\lambda_i]$ determined by the equalities (5) is among the vectors (7) and the point examined (x') of the parallelepiped H' belongs to a parallelohedron of the series (8).

It follow that

$$\int_{H'} dx_1 dx_2 \cdots dx_n \leq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n\,.$$

By making the number m grow indefinitely, one obtains

$$\Delta \leq \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

By virtue of Theorem I, it is necessary that

$$\Delta = \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

Properties of faces in $n-1$ dimensions of a parallelohedron.

12

Suppose that a parallelohedron R be defined by the independent inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0\,. \quad (k = 1, 2, \ldots, \delta)$$

Designate by $P_k (k = 1, 2, \ldots, \sigma)$ the faces in $n - 1$ dimensions of the parallelohedron R determined by the corresponding equations

$$a_{0k} + \sum a_{ik} x_i = 0 \,. \tag{1}$$

Let (α_i) be a point which is interior to the face P_k. Examine a parallelepiped K defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon \,. (i = 1, 2, \ldots, n) \tag{2}$$

One can choose a parameter ϵ however small that one will have the inequalities

$$a_{0r} + \sum a_{ir} x_i > 0, \; (r = 1, 2, \ldots, \sigma, r \neq k) \tag{3}$$

for any point (x_i) of the parallelepiped K. It results in that all the points of the parallelepiped K verifying the inequality

$$a_{0k} + \sum a_{ik} x_i \geq 0 \tag{4}$$

belong to the parallelohedron R. As the point (α_i) verifies the equation (1), the equation (4) reduces, by reason of (2), to this one here

$$\sum a_{0k} u_i \geq 0 \,.$$

I argue that one can choose a value of the parameter ϵ however small that all the points of the parallelepiped K verifying the inequality

$$\sum a_{ik} u_i \leq 0$$

will belong to another parallelohedron R_k of the set (R). By relying on the demonstrated properties of the group G of vectors, one will easily demonstrate the proposition stated.

Two parallelohedra R and R_k are contiguous by the face P_k in $n-1$ dimensions. Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron R_k to

R. The face P_k which is defined in the parallelohedron R by the equation (1) will be defined in the parallelohedron R_k by the equation

$$-a_{0k} - \sum a_{ik} x_i = 0 \,. \tag{5}$$

By carrying out a translation of the face P_k the length of the vector λ_{ik}, one obtains another face P_k of the parallelohedron R which will be within the parallelohedron determined by the equation

$$-a_{0k} - \sum a_{ik}(x_i - \lambda_{ik}) = 0 \,.$$

One will call parallel the faces P_k and P'_k of the parallelohedron R. We have arrived at the following important result:

All the faces in $n-1$ dimensions of a parallelohedron can be divided into pairs of parallel faces.

13

Designate by

$$R_1, R_2, \ldots, R_\sigma$$

all the parallellohedra which are contiguous to the parallelohedron R by the faces $P_1, P_2, \ldots, P_\sigma$. Designate by

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{i\sigma}] \tag{6}$$

the corresponding vectors.

By virtue of the definition of the parallelohedron, the vectors (6) form the basis of the group G. Among the vectors of this group there exist the systems of n vectors which form a basis of the group G.

Congruent faces in different dimensions of a parallelohedron.

14

Suppose that a face $P(\nu)$ in ν dimensions of a parallelohedron R also belongs to the parallelohedra

$R_1, R_2, \ldots, R_\tau$ of the set (R). Let (α_i) be a point which is interior to the face $P(\nu)$. One can determine a positive value of the parameter ϵ in such a manner that all the point of the parallelepiped K defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon \quad (i = 1, 2, \ldots, n)$$

belong to the parallelohedra $R, R_1, R_2, \ldots, R_\tau$.

Designate by $[\lambda_{ik}]$ the vectors the length of which one will carry out the translations of parallelohedra R_k into $R \, (k = 1, 2, \ldots, \tau)$.

By carrying out the translations of the face $P(\nu)$ the length of vectors $[\lambda_{ik}] \, (k = 1, 2, \ldots, \tau)$, one obtains the new faces

$$P'(\nu), P''(\nu), [\ldots,] P^{(\tau)}(\nu)$$

of the parallelohedron R.

Definition I. *One will call congruent the faces of the parallelohedron R*

$$P'(\nu), P''(\nu), \ldots, P^{(\tau)}(\nu)$$

in ν dimensions $(\nu = 0, 1, 2, \ldots, n-1)$.

15

Theorem. *The number of parallelohedra of the set* (R) *which are contiguous by the same face in ν dimensions can not be less than* $n + 1 - \nu \, (\nu = 0, 1, 2, \ldots, n-1)$.

Suppose that the face $P(\nu)$ be determined within the parallelohedron R by the equation

$$a_{0r} + \sum a_{ir} x_i = 0. \ (r = 1, 2, \ldots, \mu) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_\mu$ the parallelohedra which are contiguous to R by the faces in $n-1$ dimensions defined by the equations (1). The face $P(\nu)$ will belong to all the parallelohedra $R_1, R_2, \ldots, R_\mu$, therefore

$$\tau \geq \mu.$$

As the face $P(\nu)$ is in ν dimensions, it is necessary that

$$\mu \geq n - \nu,$$

and as a result

$$\tau \geq n - \nu.$$

16

Definition II. *One will call simple a face in ν dimensions which belong to only $n + 1 - \nu$ parallelohedra of the set (R).*

Definition II. *One will call primitive a parallelohedron, all the faces in different dimensions of which are simple.primitive parallelohedron*

The primitive parallelohedra possess many important properties which simplify the study.

In the subsequent studies, one will study only the primitive parallelohedron and all the nonprimitive parallelohedra which can be considered as a boundary of primitive parallelohedra.

I am inclined to think that each primitive parallelohedron can be considered in this point of view, but I have not been successful in demonstrating this.

Section II

Fundamental properties of primitive parallelohedra

Definition of primitive parallelohedra.

17

We have called in Number 16 "primitive parallelohedron" all parallelohedron, all the faces in different dimensions of which are simple.

Theorem I. *for a parallelohedron to be primitive it is necessary and sufficient that all the vertices be simple.*

The theorem stated is evident by virtue of the definition established.

Theorem II. *Two primitive parallelohedra belonging to the set (R) can be contiguous by only one face in $n-1$ dimensions.*

Suppose that a face $P(\nu)$ in ν dimensions of a primitive parallelohedron R be determined with the aid of $n-\nu$ equations

$$a_{0r} + \sum a_{ir}x_i = 0. \ (r = 1, 2, \ldots, n-\nu) \qquad (1)$$

Designate by $R_1, R_2, \ldots, R_{n-\nu}$ the parallelohedra which are contiguous to the parallelohedron R by the faces in $n-1$ dimensions defined by the aid of equations (1). The face $P(\nu)$ will not belong to the parallelohedra $R_1, R_2, \ldots, R_{n-\nu}$ by virtue of the definition established, thus the theorem introduced is demonstrated.

18 *Edges of primitive parallelohedra of the set (R)*

Let (α_i) be a vertex of the primitive parallelohedron R determined by n equations

$$a_{0k} + \sum a_{ik}x_i = 0. \ (k = 1, 2, \ldots, n) \qquad (1)$$

Designate by $R_1, R_2, \ldots, R_n$ the parallelohedra contiguous to the parallelohedron R by the faces in $n-1$ dimensions determined with the help of equation (1).

By virtue of the definition established, the vertex (α_i) will not belong to the parallelohedra $R_1, R_2, \ldots, R_n$ of the set (R).

Determine n numbers $P_{1k}, P_{2k}, \ldots, P_{nk}$ with the help of equations

$$\sum a_{ir} P_{ik} = 0. \; (r = 1, 2, \ldots, n; r \neq k; k = 1, 2, \ldots, n) \qquad (2)$$

The equations (2) do not define the number $P_{1k}, P_{2k}, \ldots, P_{nk}$ to a common factor. Attach to the equations (2) a condition

$$\sum a_{ik} p_{ik} > 0 \; (k = 1, 2, \ldots, n) \qquad (3)$$

and consider a vector g_k determined with the help of equalities

$$x_i = \alpha_i + p_{ik}\rho \text{ where } \rho \geq 0.$$

By attributing to the parameter ρ positive values sufficiently small, one will determine, by (3), the points of the vector g_k belonging to R. By putting

$$\alpha_{ik} = \alpha_i + p_{ik}\rho_k,$$

one will determine a vertex (α_{ik}) of the parallelohedron R adjacent to the vertex (α_i) by an edge $P_k(1)$ of the parallelohedron R $(k = 1, 2, \ldots, n)$. One will characterise the edge $P_k(1)$ by the symbol $[\alpha_i, \alpha_{ik}]$.

Observe that all the points of the edge $P_k(1)$ verifies $n - 1$ equations

$$a_{0r} + \sum a_{ir} x_i = 0. \quad (r = 1, 2, \ldots, n; r \neq k)$$

It follows that the edge $P_k(1)$ belongs to the parallelohedra

$$R, R_1, \ldots, R_{k-1}, R_{k+1}, \ldots, R_n \quad (k = 1, 2, \ldots, n)$$

and by virtue of the definition established, does not belong to any other parallelohedron of the set (R).

One concludes that the parallelohedra

$$R_1, R_2, \ldots, R_n$$

are contiguous by an edge too. By designating this edge by $P_0(1)$, one will determine it with the symbol $[\alpha_i, \alpha_{i0}]$ by putting

$$\alpha_{i0} = \alpha_i + p_{i0}\rho_0.$$

We have arrived at the following result:

There exist $n+1$ *edges of parallelohedra of the set* (R)*, contiguous by one common vertex of these parallelohedra.*

Observe that $n-1$ edges

$$P_1(1), \dots, P_{k-1}(1), P_{k+1}(1), \dots, P_k(1) \quad (k = 1, 2, \dots, n)$$

define a face in $n-1$ dimensions which is common to the parallelohedra R and $R_k\,(k = 1, 2, \dots, n)$. Two parallelohedra R_k and $R_h\,(k = 1, 2, \dots, n; h = 1, 2, \dots, n)$ are contiguous by a face in $n-1$ dimensions which is defined by $n-1$ edges

$$P_r(1). \quad (r = 0, 1, 2, \dots, n; r \neq k, r \neq h)$$

Canonical form of equations which define a vertex of a primitive parallelohedron.

19

By conserving the previous notations, one can determine the vertex (α_i) within the parallelohedron R with the help of equations

$$u_k(a_{0k} + \sum a_{ik}x_i) = 0, \ (k = 1, 2, \dots, n) \qquad (1)$$

$u_1, u_2, \dots, u_n$ being positive arbitrary parameters. One will say that the equation

$$-u_k(a_{0k} + \sum a_{ik}x_i) = 0 \text{ where } u_k > 0$$

does not define within the parallelohedron R a face in $n-1$ dimensions because the inequality

$$-u_k(a_{0k} + \sum a_{ik}x_i) \geq 0$$

will not satisfy all the points of the parallelohedron R.

Theorem. *One can determine the positive values of parameters* $u_1, u_2, \ldots, u_n$ *to a common factor, such that by putting*

$$a'_{0k} = u_k a_{0k},\, a'_{ik} = u_k a_{ik},\ (i = 1, 2, \ldots, n; k = 1, 2, \ldots, n)$$

one will define the vertex (α_i) *within the parallelohedron* R *by the equations*

$$\sum a'_{ik}(x_i - \alpha_i) = 0,\ (k = 1, 2, \ldots, n) \tag{2}$$

and one will define the vertex (α_i) *within the parallelohedron* R_k $(k = 1, 2, \ldots, n)$ *by the equations*

$$\left\{ \begin{array}{l} \sum(a'_{ih} - a'_{ik})(x_i - \alpha_i) = 0,\ (h = 1, 2, \ldots, n; h \neq k) \\ -\sum a'_{ik}(x_i - \alpha_i) = 0.\ (k = 1, 2, \ldots, n) \end{array} \right. \tag{3}$$

Take an arbitrary positive parameter δ and determine the parameters $u_1, u_2, \ldots, u_n$ after the equations

$$u_k \sum a_{ik}(\alpha_i - \alpha_{i0}) = \delta.\ (k = 1, 2, \ldots, n) \tag{4}$$

I argue that the values $u_1, u_2, \ldots, u_n$ obtained satisfy the conditions of the theory stated.

To demonstrate this, observe in the first place that the equations (4) define the positive values of $u_1, u_2, \ldots$, u_n. In effect, we have seen in Number 18 that the edge $P_0(1)$ defined by the equalities

$$x_i = \alpha + u(\alpha_{i0} - \alpha_i) \text{ where } 0 \leq u \leq 1 \tag{5}$$

does not belong to the parallelohedra $R_1, R_2, \ldots, R_n$. One concludes that by attributing to the parameter u any negative values sufficiently small, one will determine by the equality (5) a point which will be interior to the parallelohedron R. It follows that

$$\sum a_{ik}(\alpha_i - \alpha_{i0}) > 0, \quad (k = 1, 2, \ldots, n)$$

and the equations (4) give

$$u_k > 0. \quad (k = 1, 2, \ldots, n)$$

This established, designate by

$$\sum a_{ir}^{(k)}(x_i - \alpha_i) = 0 \ (r = 1, 2, \ldots, n) \tag{6}$$

the equations which define the vertex (α_i) in the parallelohedron $R_k \, (k = 1, 2, \ldots, n)$

Observe that n edges $P_r(1) \, (r = 0, 1, 2, \ldots, n, r \neq k)$ are contiguous by the vertex (α_i) in the parallelohedron R_k. Each equation (6) will be verified by $n - 1$ edges. One can thus put

$$\left\{ \begin{array}{l} \sum a_{ik}^{(k)}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \ldots, n; r \neq k) \\ \sum a_{ik}^{(k)}(\alpha_{i0} - \alpha_i) > 0 \end{array} \right. \tag{7}$$

and

$$\left\{ \begin{array}{l} \alpha a_{ih}^{(k)}(\alpha_{ir} - \alpha_i) = 0, \ (r = 0, 1, 2, \ldots, n; r \neq k, r \neq h) \\ \sum a_{ih}^{(k)}(\alpha_{ih} - \alpha_i) > 0. \ (h = 1, 2, \ldots, n; h \neq k) \end{array} \right. \tag{8}$$

The conditions established define the coefficients of equations (6) to a common positive factor, which can be arbitrarily chosen.

Observe that the coefficients of equations (1), which define the vertex (α_i) in the parallelohedron R are also determined to a common positive factor and satisfy the conditions

$$\left\{ \begin{array}{l} \sum a_{ik}(\alpha_{ir} - \alpha_i) = 0, \\ \qquad (r = 1, 2, \ldots, n; r \neq k; k = 1, 2, \ldots, n) \\ \sum a_{ik}(\alpha_{ik} - \alpha_i) > 0 \end{array} \right. \tag{9}$$

The equalities (4), (7), (8) and (9), one takes

$$\left. \begin{array}{l} a_{ik}^{(k)} = -\delta_k u_k a_{ik}, \\ a_{ih}^{(k)} = \delta_h (u_h a_{ih} - u_k a_{ik}), \end{array} \right\}$$

$$(i = 1, 2, \ldots, n; h = 1, 2, \ldots, n; h \neq k)$$

where $\delta_1, \delta_2, \ldots, \delta_n$ are positive factors. One can put

$$\delta_1 = 1, \delta_2 = 1, \ldots, \delta_n = 1,$$

and the equations (6) become

$$\sum(u_h a_{ih} - u_k a_{ik})(x_i - \alpha_i) = 0, \; (h = 1, 2, \ldots, n; h \neq k)$$
$$-\sum u_k a_{ik}(x_i - \alpha_i) = 0.$$

The theorem introduced is thus demonstrated.

One will say that the equations (2) and (3) which define the vertex (α_i) in the contiguous parallelohedron $R, R_1, \ldots, R_n$ are presented in the canonical form.

We have seen in Number 18 that the parallelohedra R_k and $R_h\,(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n)$ are contiguous by a face in $n-1$ dimensions. As this face is characterised by the edges $P_r(1)\,(r = 0, 1, 2, \ldots, n; r \neq k; r \neq h)$, one will determine it in the parallelohedron R_h by the canonical equation

$$\sum(a'_{ik} - a'_{ih})(x_i - \alpha_i) = 0.$$

Canonical form of inequalities which define a positive parallelohedron.

20

Suppose that a primitive parallelohedron R is determined with the help of independent inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Bu designating with $u_1, u_2, \ldots, u_\sigma$ of arbitrary positive parameters, one will determine the parallelohedron R with the help of independent inequalities

$$u_k(a_{0k} + \sum a_{ik} x_i) \geq 0. \; (k = 1, 2, \ldots, \sigma) \qquad (1)$$

We will see how all the problem of the study of primitive parallelohedra comes down to the appropriate choice of parameters $u_1, u_2, \ldots, u_\sigma$.

Fundamental Theorem. One can determine the positive values of parameters $u_1, u_2, \ldots, u_\sigma$ *to a common factor, such that by putting*

$$a'_{0k} = u_k a_{0k}, a'_{ik} - u_k a_{ik}, \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, \sigma)$$

one will determine the parallelohedron R *with the help of inequalities*

$$a'_{0k} + \sum a'_{ik} x_i \geq 0 \ (k = 1, 2, \ldots, \sigma) \tag{2}$$

which enjoy the following property: all the vertices of the parallelohedron R *will be determined by the equations presented in the canonical form.*

One will call the inequalities (2) canonical.

By conserving the previous notations, suppose that one had chosen the parameters $u_1, u_2, \ldots, u_n$ in such a manner that the vertex (α_i) is determined by the canonical equations

$$a'_{0k} + \sum a'_{ik} x_i = 0. \quad (k = 1, 2, \ldots, n)$$

Examine the equations which define a vertex (α_{ik}) $(k = 1, 2, \ldots, n)$ of the parallelohedron R adjacent to the vertex (α_i) by the edge $P_k(1)$.

The vertex (α_{ik}) satisfies $n - 1$ equations

$$a'_{0h} + \sum a'_{ih} x_i = 0. \ (h = 1, 2, \ldots, n; h \neq k) \tag{3}$$

Designate by

$$b_{0k} + \sum b_{ik} x_i = 0 \tag{4}$$

the n^{th} equation which defines the vertex (α_{ik}).

Determine the positive parameters v_h $(h = 1, 2, \ldots, n,$ $h \neq k)$ and v_k corresponding to the equations (3) and (4), which reduces to these equations in the canonical form:

$$v_h(a'_{0h} + \sum a'_{ih} x_i) = 0 \quad (h = 1, 2, \ldots, n; h \neq k)$$

and

$$v_k(b_{0k} + \sum b_{ik}x_i) = 0.$$

I argue that one can put

$$v_h = 1. \quad (h = 1, 2, \ldots, n; h \neq k)$$

To demonstrate this, examine the canonical equation which defines in the parallelohedron $R_h\,(h = 1, 2, \ldots, n;$ $h \neq k)$ a face in $n-1$ dimensions common to the parallelohedra R_r and $R_h\,(r = 1, 2, \ldots, n; r \neq h, r \neq, k)$.

By virtue of the theorem of Number 19, this face will be determined within R_h by the canonical equation

$$\sum(a'_{ir} - a'_{ih})(x_i - \alpha_i) = 0.$$

Besides, this same face will be determined in R_h, by virtue of the supposition made, by the canonical equation

$$\sum(v_r a'_{ir} - v_h a'_{ih})(x_i - \alpha_i) = 0.$$

It results in that

$$v_r a'_{ir} - v_h a'_{ih} = \delta(a'_{ir} - a'_{ih}), \quad (i = 1, 2, \ldots, n)$$

and so

$$v_r = \delta, v_h = \delta,$$

thus

$$v_r = v_h. \quad (r = 1, 2, \ldots, n; r \neq k; r \neq h)$$

As the parameters $v_h\,(h = 1, 2, \ldots, n)$ are defined to a factor, one can put

$$v_h = 1, \quad (h = 1, 2, \ldots, n; h \neq k)$$

and it only remains to determine the parameter v_k in order to define the vertex (α_{ik}) by the canonical equations. 21

By applying the procedure explained to all the vertices of the parallelohedron R adjacent to the vertex (α_i) and so on, one will successively determine the values of various parameters corresponding to all the inequalities (1).

It can turn out that one determines for one inequality the value of the corresponding parameter in various manners. I argue that all these values of the same parameter coincide.

The problem posed is extremely difficult. It is within this group of studies explained that is manifested their true geometrical characteristic, and one does not manage to master the difficulties which arise as a result with the help of geometrical methods.

Set of simplexes corresponding to the various vertices of a primitive parallelohedron.

22

We have seen in Number 4 that the various vertices of a parallelohedron R $(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$ correspond to the domains

$$A_1, A_2, \ldots, A_s \tag{1}$$

which uniformly fill the space in n dimensions.

By conserving the previous notations, examine a domain A which corresponds to the vertex (α_i) of the parallelohedron R.

The domain A is composed of points determined by the equalities

$$x_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, n) \tag{2}$$

The domain A possesses n faces in $n-1$ dimensions which correspond to n edges $P_k(1), (k = 1, 2, \ldots, n)$ contiguous by the vertex (α_i).

One will call the domain A simple.

Extract from the domain A a simplex L by the solution with the help of equalities

$$x_i = \sum_{k=1}^{n} \vartheta_k u_k a_{ik} \text{ where } \sum \vartheta_k \leq 1 \text{ and } \vartheta_k \geq 0.$$

$$(k = 1, 2, \dots, n)$$

The simplex L possesses $n+1$ faces in $n-1$ dimensions which are opposite to $n+1$ vertices

$$(0), (u_1 a_{i1}), (u_2 a_{i2}), \dots, (u_n a_{in}),$$

Examine the face of the simplex L which is opposite to the vertex (0). One can present the equation which defines this face in the form

$$1 - \sum p_i x_i = 0. \qquad (3)$$

It follows that one will have an inequality

$$1 - \sum p_i x_i > 0$$

for any point of L which does not belong to the face examined.

As the vertices of L: $(u_k a_{ik})(k = 1, 2, \dots, n)$ satisfy the equation (3), one has

$$\sum p_i a_{ik} = \frac{1}{u_k}, \ (k = 1, 2, \dots, n) \qquad (4)$$

thus

$$\sum p_i a_{ik} > 0. \quad (k = 1, 2, \dots, n).$$

By virtue of (2), one obtains the inequality

$$\sum p_i x_i > 0$$

which holds for any point (x_i) of the domain A, the vertex (0) being excluded.

23

Examine in the same manner n domains $A_1, A_2, \dots, A_n$ which are contiguous to the domain A by faces in $n-1$ dimensions.

One will take from the simple domain A_k $(k = 1, 2, \dots, n)$ defined by the equalities

$$x_i = \sum \rho_h a_{ih} + \rho_k b_{ik} \text{ where } \rho_k \geq 0 \text{ and } \rho_h \geq 0,$$

$$(h = 1, 2, \dots, n; h \neq k)$$

a simplex L_k composed of points

$$x_i = \sum \vartheta_h u_h a_{ih} + \vartheta_k v_k b_{ik} \text{ where } \sum \vartheta_h + \vartheta_k \leq 1, \vartheta_k \geq 0, \vartheta_h \geq 0$$
$$(h = 1, 2, \dots, n; h \neq k)$$

Designate by

$$1 - \sum p_{ik} x_i = 0$$

the equation of the face of the simplex L_k which is opposite to the vertex (0).

One will have the equalities

$$\sum p_{ik} a_{ih} = \frac{1}{u_b} \quad (h = 1, 2, \dots, n; h \neq k)$$

and

$$\sum p_{ik} b_{ih} = \frac{1}{u_k}.$$

By virtue of equalities (4), one obtains

$$\sum p_{ik} a_{ih} = \sum p_i a_{ih}. \quad (h = 1, 2, \dots, n; h \neq k)$$

It follows that

$$\sum p_{ik} x_i = \sum p_i x_i, \tag{5}$$

for any point (x_i) belonging to the face common to domains A and A_k.

By applying the procedure explained to the domains which are contiguous to the domains $A_1, A_2, \ldots, A_n$ and so on, one will extract from any domain of the set (1) a corresponding simplex.

It can turn out that one extracts from the same domain the corresponding simplex by various manners. I argue that all these simplexes coincide.

It is clear that the problem stated does not differ from a formulation of the problem put forward in Number 21.

We shall show a new formulation of this problem.

On a function defined by the set of simplexes corresponding to the various vertices of a primitive parallelohedron.

24

Introduce within our study a function $P(x_1, x_2, \ldots, x_n)$ of variables $x_1, x_2, \ldots, x_n$ by defining as follows.

1. One will determine the function $P(x_1, x_2, \ldots, x_n)$ in the domain A by the formula

$$P_{(A)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i.$$

2. In the domains A_k $(k = 1, 2, \ldots, n)$ contiguous to the domain A by the faces in $n-1$ dimensions, one will determine the function $P(x_1, x_2, \ldots, x_n)$ by the formula

$$P_{(A_k)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} P_{ik} x_i. \quad (k = 1, 2, \ldots, n)$$

Let

$$A, A', A'', \ldots, A^{(m)} \tag{1}$$

be a series of domains which are successively contiguous by the faces in $n-1$ dimensions. One will successively take from these domains the following simplexes.

$$L, L', L'', \ldots, L^{(m)}$$

and one will determine the corresponding function.

$$\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i' x_i, \sum_{i=1}^{n} p_i'' x_i, \ldots, \sum_{i=1}^{n} p_i^{(m)} x_i.$$

One will define the function $P(x_1, x_2, \ldots, x_n)$ in the domains (1) by the formula

$$P_{(A^{(k)})}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i^k x_i. \quad (k = 1, 2, \ldots, m)$$

Fundamental Theorem. *The function $P(x_1, x_2, \ldots, x_n)$ defined by the conditions 1, 2, and 3 is continuous and uniform in all the space in n dimensions.*

Observe that the fundamental introduced only give as a new formulation of the fundamental theorem of Number 20.

Take an arbitrary closed contour C. By traversing the contour C, one can determine a series of domains successively contiguous by faces in $n-1$ dimensions in which belong the points of the contour C:

$$A^{(0)}, A', A'', \ldots, A^{(m)}, A^{(0)}, A', \ldots$$

To demonstrate this, take a point (ξ_{i0}) of the contour C and designate by C_0 a curve which is being traversed within a domain $A^{(0)}$ leaving from the initial point (ξ_{i0}). Suppose that the curve C_0 does not coincide with the contour C and designate by (ξ_{i1}) the final point of the curve C_0.

Suppose that on leaving the point (ξ_i) one got out of the domain $A^{(0)}$ and that one entered inside the domain

A'. Designate by C_1 a group of contour C which one has traversed in the domain A' when leaving the point (ξ_{i1}) and so on and so forth. Suppose that one had divided with the help of the procedure described the contour C into groups

$$C_0, C_1, \ldots, C_m, C_0$$

which belong to the domains

$$A^{(0)}, A', A'', \ldots, A^{(m)}, A^{(0)}. \tag{2}$$

It can turn out that two adjacent domains of this series $A^{(k)}$ and $A^{(k+1)}$ are not contiguous by a face in $n-1$ dimensions. One inserts in this case between the domains $A^{(k)}$ and $A^{(k+1)}$ new domains of the solutions as follows:

A point $(\xi_{i,k+1})$ which is the final point of the curve C_k and which gives the initial point of the curve C_{k+1} belongs, by virtue of the supposition made, to the domains $A^{(k)}$ and $A^{(k+1)}$. One concludes that the point $(\xi_{i,k+1})$ is interior to a face $A^{(k)}(\nu)$ in ν dimensions which is common to the domains $A^{(k)}$ and $A^{(k+1)}$.

One can determine a parallelepiped K with the help of equalities

$$x_i = \xi_{i,k+1} + u_i \text{ where } |u_i| \leq \epsilon, \ (i = 1, 2, \ldots, n)$$

of manner such that the points of the parallelepiped D do not belong to the domains of the series (1) (Number 22) which are contiguous by the face $A^{(k)}(\nu)$.

Take with the parallelepiped K two points (x_{ik}) and $x_{i,k+1}$ which ae interior to the domain $A^{(k)}$ and $A^{(k+1)}$ and take within the parallelepiped K a curve $C^{(k)}$ which joins the points (x_{ik}) and $(x_{i,k+1})$. One can choose this curve in such a manner that it does not pass beyond any face of domains (1) (Number 22) of which the number of dimension s is less than $n-1$.

Suppose that the curve $C^{(k)}$ traverse the domain

$$A^{(k)}, A_1^{(k)}, \ldots, A_\mu^{(k)}, A^{(k+1)}.$$

By virtue of the supposition made, the domains obtained are successively contiguous by the faces in $n-1$ dimensions. All these domains are contiguous pairwisely by the face $A^{(k)}(\nu)$.

In the same manner, one will examine all the pairs of adjacent domains of the series (2) and one will form the series

$$A^{(0)}, A', \ldots, A^{(m)}, A^{(0)}$$

of domains successively contiguous by the faces in $n-1$ dimensions to which belong all the points of the closed contour C given.

25

This established, observe that the fundamental theorem introduced is true in the case where all the domains (2) are contiguous in at least one edge.

In effect, suppose that one had successively taken away from the domains (2) the simplexes

$$L^{(0)}L', L'', \ldots, L^{(m)}, L^{(m+1)} \tag{3}$$

I argue that the simplex $L^{(m+1)}$ taken from the domain $A^{(0)}$ coincide with the simplex $L^{(0)}$. To demonstrate this, designate by

$$\sum p_i^{(m+1)} x_i = \delta \sum p_i^{(0)} x_i \, . \tag{4}$$

By virtue of the supposition made, the domains (2) are contiguous by at least one edge. Let (a_i) be a point of this edge.

As the domains $A^{(0)}$ and A' are contiguous by a face in $n-1$ dimensions, one will have, as we have seen this in Number 23, an equality

$$\sum p_i^{(0)} x_i = \sum p_i' x_i \tag{5}$$

which holds for any point (x_i) of thee face common to the domains $A^{(0)}$ and A'.

By making $x_i = a_i$, one obtains

$$\sum p^{(0)} a_i = \sum p'_i a_i$$

In the same manner, one will obtain

$$\sum p_i^{(0)} a_i = \sum p'_i a_i = \ldots \sum p_i^{(m)} a_i = \sum p_i^{(m+1)} a_i\,.$$

On the other hand, the identity (4) gives

$$\sum p_i^{(0)} a_i = \delta \sum p_i^{(m+1)} a_i\,,$$

and as $\sum p_i^{(0)} a_i > 0$, then $\delta = 1$, therefore

$$\sum p_i^{(m+1)} x_i = \sum p_i^{(0)} x_i$$

and the two simplexes $L^{(0)}$ and $L^{(m+1)}$ coincide.

By virtue of the definition established, one will determine the function $P(x_1, \ldots, x_n)$ in the domain $A^{(0)}$ by the formula

$$p_{(A^0)}(x_1, x_2, \ldots, x_n) = \sum p_i^{(}0) x_i$$

by leaving the domain $A^{(0)}$ and by returning to within that domain after having traversed the path C.

26

We will see that the general case can be brought back to the case examined. To this effect, suppose the projection of any one contour C evaluated in relation to surface S is determined by the equation

$$\sum x_i^2 = 1.$$

By putting

$$x'_i = \frac{x_i}{\sqrt{\sum x_i^2}},\ (i = 1, 2, \ldots, n) \qquad (6)$$

one will call the point (x'_i) the projection of the point (x_i) within the surface S.

Designate by C' a projection of this contour C.

Suppose that by traversing the contour C', one returns to the initial point (ξ'_i) with the same solution of the function $P(x_1, \ldots, x_n)$ of which when leaving that point. I argue that one will return to the corresponding point ξ_i of the contour C with the same solution of the function $P(x_1, \ldots, x_n)$.

To demonstrate this, it suffices to observe that the points (ξ_i) and (ξ'_i), by virtue of equalities (6), belong to the same domains of the series (2).

One concludes that it suffices to examine the different closed contours belonging to the surface S.

27

Introduce in our study a function $d(x_i, x'_i)$ being defined by the formula

$$d(x_i, x'_i) = \sqrt{\sum (x'_i - x_i)^2}.$$

One will call distance between two points (x_i) and (x'_i) the corresponding value of the function $d(x_i, x'_i)$.

Lemma. *One can determine a positive parameter δ satisfying the following condition: every closed contour C belonging to the surface S will be situated in the domains which are contiguous by at least one edge, if the distance of all the point of the contour C, each of all to the rest, do not exceed the limit δ.*

Let (ξ_i) be a point of the contour C belonging to the domain A. Put

$$\xi_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, n)$$

By virtue of the equation

$$\sum \xi_i^2 = 1,$$

the sum $\sum_{k=1}^{n} \rho_k$ is not less than a positive fixed limit.

$$\sum_{k=1}^{n} \rho_k \geq \tau. \tag{7}$$

Suppose that the contour C is not situated entirely within the domain A.

Let (ξ_i') be a point of C which does not belong to the domain A. By putting

$$\xi_i' = \sum_{k=1}^{n} \rho_k' a_{ik}, \tag{8}$$

one will have among the numbers $\rho_1', \rho_2', \ldots, \rho_n'$ at least one negative number.

Suppose, to fix an ideas, that

$$\rho_1' \geq 0, \rho_2' \geq 0, \ldots, \rho_\mu' \geq 0 \tag{9}$$

and that

$$\rho_{\mu+1}' < 0, \rho_{\mu+2}' < 0, \ldots, \rho_n' < 0. \tag{10}$$

After the supposition made, one has the inequality

$$d(\xi_i, \xi_i') \geq \delta.$$

One can choose the parameter δ, of such a manner that one had the inequalities

$$|\rho_k' - \rho_k| < \epsilon, \ (k = 1, 2, \ldots, n) \tag{11}$$

ϵ being a positive parameter also small as one would wish.

By (10), one obtains

$$0 \leq \rho_k < \epsilon, -\epsilon < \rho_k' < 0. \ (k = \mu + 1; \mu + 2, \ldots, n) \tag{12}$$

Choose among the numbers $\rho_1, \rho_2, \ldots, \rho_n$ the one which is the largest. By virtue of the inequality (7), this number can not be less than $\frac{\tau}{n}$. By supposing that

$$\epsilon < \frac{\tau}{n},$$

one will find the number looked for among the numbers $\rho_1, \rho_2, \ldots, \rho_\mu$. Suppose, to fix the ideas, that

$$\rho_1 > \frac{\tau}{n}.$$

the inequality (11) gives

$$\rho_1' > \frac{\tau}{n} - \epsilon. \tag{13}$$

This posed, suppose that the point (ξ_i') belonged to the domain A' and put

$$\xi_i' = \sum_{k=1}^{n} u_k a_{ik}' \text{ where } u_k \geq 0. \ (k = 1, 2, \ldots, n) \tag{14}$$

Designate by (α_i) and α_i' two vertices of the parallelohedron R corresponding to domains A and A' by defining them by the equations

$$a_{0k} + \sum a_{ik} x_i = 0, \quad (k = 1, 2, \ldots, n)$$

and by the equations

$$a_{0k}' + \sum a_{ik}' x_i = 0. \ (k = 1, 2, \ldots, n) \tag{15}$$

By virtue of equality (8) and (14), one obtains an identity

$$\rho_0' + \sum_{k=1}^{n} \rho_k' (a_{0k} + \sum a_{ik} x_i) = \sum_{k=1}^{n} u_k (a_{0k}' + \sum a_{ik}' x_i). \tag{16}$$

By making in this identity $x_i' = \alpha_i$, one finds

$$\rho_0' = \sum u_k (a_{0k}' + \sum a_{0k}' \alpha_i) \geq 0. \tag{17}$$

By making in the identity (16) $x_i = \alpha'_i$, it will become

$$\rho'_0 + \sum_{k=1}^{n} \rho'_k (a_{0k} + \sum a_{ik}\alpha'_i) = 0. \tag{18}$$

Suppose that

$$a_{01} + \sum a_{i1}\alpha'_i > 0.$$

By virtue of (7), (12), (13) and (17), one will have

$$\rho'_0 + \sum_{k=1}^{\mu} \rho'_k (a_{0k} + \sum a_{ik}\alpha'_i) > (\frac{\tau}{n} - \epsilon)(a_{01} + \sum a_{i1}\alpha'_i),$$

$$\sum_{k=\mu+1}^{n} \rho'_k (a_{0k} + \sum a_{ik}\alpha'_i) \geq -\epsilon \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha'_i),$$

and the equality (18) gives

$$\frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha'_i) <$$

$$\epsilon \left[a_{01} + \sum a_{i1}\alpha'_i + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha'_i) \right]. \tag{19}$$

Designate

$$A = \frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha'_i) \text{ and } B =$$

$$a_{01} + \sum a_{i1}\alpha'_i + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha'_i),$$

one will have

$$A > 0 \text{ and } B > 0,$$

and as a result

$$\epsilon > \frac{A}{B}. \tag{20}$$

One could determine the ratio $\frac{A}{B}$ correspondent to the different vertices of the parallelohedron R. Designate by ω the smallest of these ratios which is not zero. The parameter ϵ being arbitrary, one can suppose that

$$\epsilon \leq \omega.$$

The inequality (20) becomes impossible, it is therefore necessary that $A = 0$ or [to put it] differently

$$a_{01} + \sum a_{i1}\alpha'_i = 0.$$

By virtue of the equality obtained, the coefficients of the equation

$$a_{01} + \sum a_{i1}x_i = 0$$

are proportional to those of an equation which is among the equations (15).

By putting

$$a_{01} + \sum a_{i1}x_i = u(a'_{0h} + \sum a'_{ih}x_i) \text{ where } u > 0,$$

one will have

$$a_{i1} = ua'_{ih}. \quad (i = 1, 2, \dots, n)$$

We have arrived at the following result: all the domain traversed by the contour C examined are contiguous by at least one edge which is characterised by the point (a_{i1}).

28

We are now in the state of reaching the demonstration of the fundamental theorem announced.

Let C be any contour belonging to the surface S. Suppose that on leaving the point (ξ_i) one passes via the points $(\xi_i^{(0)}), (\xi'_i), (\xi''_i)$ and one returns to the point (ξ_i).

The path around the contour C can be replaced by the paths $C^{(0)}$ and C'.

The contour $C^{(0)}$ will be composed of a group $(\xi_i) - (\xi_i^{(0)})$ of C, of the vector $[\xi_i^{(0)}, \xi''_i]$ and of a group $(\xi_i^{(0)}) - (\xi'_i) - (\xi''_i)$ of the contour C and of the vector $[\xi''_i, \xi_i^{(0)}]$.

Suppose that by traversing the paths $C^{(0)}$ and C' one uniformly defined the function $P(x_1, x_2, \ldots, x_n)$. In this case the trajectory by the group $(\xi_i^{(0)})-(\xi_i')$ of the contour C can be replace by the path the length of the vector $[\xi_i^{(0)}, \xi_i'']$.

By replacing the group $(\xi_i^{(0)})-(\xi_i')-(\xi_i'')$ of the contour C by the vector $[\xi_i^{(0)}, \xi_i'']$, one will transform the contour C to $C^{(0)}$, thus, by traversing the contour C, one will return to the point (ξ_i), by virtue of suppositions made, with the same solution of the function $P(x_1, x_2, \ldots, x_n)$.

Two contours $C^{(0)}$ and C' can be examined in the same manner and so on.

Suppose that one had determined the contours

$$C_1, C_2, \ldots, C_m \tag{21}$$

which replace the path C. By supposing that the function $P(x_1, x_2, \ldots, x_n)$ be uniform the length of contour (21), one will demonstrate that it will be uniform the length of the contour C given.

This established, observe that we can always choose the contours (21), of such a manner that their contours satisfy the conditions of the lemma of the previous Number. In this case, any contour (21) will be situated within domains which are contiguous by at least one edge. We have seen in Number 25 that by traversing the same contours one will always return to the point of departure by the same solution of the function $P(x_1, x_2, \ldots, x_n)$ as while leaving this point. It is thus demonstrated that any closed contour C possesses the same property.

We have demonstrated that the function $P(x_1, x_2, \ldots, x_n)$ is uniformly defined in any domain of the set (1) (Number 22). It remains to demonstrate that the function $P(x_1, x_2, \ldots, x_n)$ is well defined in any point of the space in n dimensions.

Suppose that a point ξ_i belongs to two domains A and $A^{(0)}$.

I argue that the function $P(x_1, x_2, \ldots, x_n)$ for the point ξ_i will have one same value in the domain A and in the domain $A^{(0)}$.

To demonstrate this, one will form a series of domains

$$A, A', \ldots, A^{(m)}, A^{(0)}$$

which are successively contiguous by faces in $n-1$ dimensions and in which belongs the point ξ_i.

As the point ξ_i belongs to the face common to domains A and A', one will have ,by virtue of the formula (5) of Number 23,

$$P_{(A)}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A')}(\xi_1, \xi_2, \ldots, \xi_n).$$

In the same manner, one obtains

$$P_{(A')}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A'')}(\xi_1, \xi_2, \ldots, \xi_n),$$
$$\ldots$$
$$P_{(A^{(m)})}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \ldots, \xi_n).$$

It results in that

$$P_{(A)}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \ldots, \xi_n).$$

The fundamental theorem announced is thus demonstrated.

29 *Canonical form of inequalities which define the set (R) of primitive parallelohedron S.*

Choose within the set (R) of primitive parallelohedra any parallelohedron R_0 . Suppose that the parallelohedron R_0 is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Observe that we can replace these inequalities by the following canonical inequalities:

$$u(a_{0k} + \sum a_{ik}x_i) \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

u being a positive arbitrary parameter.

Designate by $R_k\,(k = 1, 2, \ldots, \sigma)$ the parallelohedron which is contiguous to the parallelohedron R_0 by the face determined within R_0 by the equation

$$a_{0k} + \sum a_{ik}x_i = 0 \tag{1}$$

and suppose that the vector $[\lambda_{ik}]$ defined a translation of the parallelohedron R_k to R_0.

It follows that the parallelohedron $R_k$will be determined by the canonical inequalities

$$a_{0h} + \sum a_{ih}(x_i + \lambda_{ik}) \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

or by the canonical inequalities

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] \geq 0, \; (h = 1, 2, \ldots, \sigma) \tag{2}$$

u_k being an arbitrary positive parameter.

The face P_k in $n-1$ dimensions common to the parallelohedra R and R_k is defined in the parallelohedron R_0 by the equation (1). Within the parallelohedron R_k, the face P_k will be determined by an equation in which the coefficients are proportional to those of the equation

$$-a_{0k} - \sum a_{ik}x_i = 0.$$

One can choose the positive parameter u_k, of a manner such that one had the identity

$$-a_{0k} - \sum a_{ik}x_i = u_k(a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})).$$

In this case, the inequality

$$-a_{0k} - \sum a_{ik}x_i \geq 0$$

is found among the inequalities (2) which define the parallelohedron R_k.

One will say that these inequalities are represented in the canonical form.

30

Observe an important property of canonical inequalities which define the parallelohedra $R_0, R_1, R_2, \ldots, R_\sigma$.

Let (α_i) be a vertex of the parallelohedron R_0 determined by the canonical equations

$$a_{0k} + \sum a_{ik} x_i = 0. \ (k = 1, 2, \ldots, n) \tag{3}$$

Examine the canonical equations which define the vertex (α_i) in the parallelohedron $R_k \, (k = 1, 2, \ldots, n)$.

The equations (3) being canonical, one will determine the vertex (α_i) within the parallelohedron R_k, by virtue of the theorem of Number 19, by the equation

$$\begin{gathered}\sum (a_{ih} - a_{ik})(x_i - \alpha_i) = 0, \quad (h = 1, 2, \ldots, n, h \neq k) \\ -\sum a_{ik}(x_i - \alpha_i) = 0 \quad (k = 1, 2, \ldots, n)\end{gathered}$$

By virtue of the supposition made, the inequality

$$-\sum a_{ik}(x_i - \alpha_i) \geq 0$$

exists among the canonical inequalities (2) which define the parallelohedron R_k, which results in that the inequalities

$$\sum (a_{ih} - a_{ik})(x_i - \alpha_i) \geq 0, \quad (h = 1, 2, \ldots, n, h \neq k)$$

also exist among the canonical inequalities (2).

One concludes that the canonical equation

$$\sum (a_{ih} - a_{ik})(x_i - \alpha_i) = 0$$

define in the parallelohedron R_k a face in $n-1$ dimensions which is common to the parallelohedra R_k and R_h. the same face will be determined in the parallelohedron R_h by a canonical equation

$$\sum (a_{ik} - a_{ih})(x_i - \alpha_i) = 0.$$

31

by applying the procedure explained, one can determine the canonical inequalities which define the parallelohedra contiguous to the parallelohedra $R_1, R_2, \ldots, R_\sigma$ and so on.

For every parallelohedron R of the set (R), one can form a series of parallelohedra

$$R_0, R', R'', \ldots, R^{(m)}, R$$

which are successively contiguous. One will determine successively the canonical inequalities that define the parallelohedra of this series.

One could arrive at the parallelohedron R by other ways and determine the canonical inequalities which define the parallelohedron R in various manners.

We shall se that the canonical inequalities which define a parallelohedron of the set (R) do not depend on the path by which one arrives at the parallelohedron (R) leaving from the principal parallelohedron R_0.

Generatrix function of the set (R) *of primitive parallelohedra.*

32

Consider a set (R) of primitive parallelohedra. Suppose any parallelohedron R of the set (R) be characterised by a vector $[\lambda_i]$ which defines a translation of parallelohedra R to a principal parallelohedron R_0.

Designate by G the group of vectors $[\lambda_i]$ which correspond to the different parallelohedra of the set (R).

Introduce in our study a function

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$$

of variables $x_1, x_2, \ldots, x_n$ and parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ by defining it within the space in n dimensions and for the group G such that:

1. Within the principal parallelohedron R_0, one will write

$$V(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0,) = 0.$$

2. Within the parallelohedron R_k which is contiguous to R_0, one will write

$$V(x_1, x_2, \ldots, x_n, \lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk} = a_{0k} + \sum a_{ik} x_i,$$

$$(k = 1, 2, \ldots, \sigma)$$

providing that in the parallelohedron R_0 the canonical equation

$$a_{0k} + \sum a_{ik} x_i = 0$$

had the face in $n - 1$ dimensions common to the parallelohedra R_0 and R_k.

3. By supposing that the parallelohedra R and R' characterised by the vectors $[\lambda_i]$ and λ'_i are contiguous by a face in $n - 1$ dimensions which is defined within R by a canonical equation

$$a_0 + \sum a_i x_i = 0,$$

one will write

$$V(x_1, x_2, \ldots, x_n, \lambda_1,', \lambda'_2, \ldots, \lambda'_n) =$$
$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) + a_0 + \sum a_i x_i.$$

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Let R be any parallelohedron of the set (R) characterised by a vector $[\lambda_i]$. One will form a series of parallelohedra.

$$R_0, R', \ldots, R^{(m)}, R$$

which are successively contiguous by faces in $n-1$ dimensions. Designate by

$$a_0^{(0)} + \sum a_i^{(0)} x_i = 0$$

the equation of the faces common to the parallelohedra R_0 and R' and defined in R_0; designate by

$$a_0' + \sum a_i' x_i = 0$$

the equation of the face common to the parallelohedra R' and R'' defined in R' and so on.

By applying the definition established, one will determine the function

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$$

by the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

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Fundamental theorem. *The function* $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ *is well defined for any vector* $[\lambda_i]$ *of the group* G.

Suppose that one had formed a series of parallelohedra

$$R, R', R'', \ldots, R^{(m)}, R \qquad (1)$$

which are successively contiguous. On leaving the parallelohedron R with any solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$, one will return inside the parallelohedron R after having traversed the parallelohedra (1) with a solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ which, by virtue of the definition established, is expressed by the vertex

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) + \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

We shall demonstrate that one will always have

$$\sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

Examine, in the first place, the case where all the parallelohedra (1) are contiguous by at least one vertex (α_i). By virtue of Theorem II of Number 17, all the primitive parallelohedra (1) will be in this case contiguous one to one through faces in $n-1$ dimensions.

Designate by

$$a_{0k} + \sum a_{ik}(x_i - \alpha_i) = 0, \qquad (k = 1, 2, \dots, m)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and $R (k = 1, 2, \dots, m)$ defined within the parallelohedron R.

We have seen in Number 30 that the canonical equation of the face common to the parallelohedra R' and R'' and defined within R' will be

$$\sum (a_{i2} - a_{i1})(x_i - \alpha_i) = 0$$

and so on and so forth. One obtains the formulae

$$\begin{aligned} a_0^{(0)} + \sum a_i^{(0)} x_i &= \sum a_{i1}(x_i - \alpha_i), \\ a_0' + \sum a_i' x_i &= \sum (a_{i2} - a_{i1})(x_i - \alpha_i), \\ &\dots \\ a_0^{(m-1)} + \sum a_i^{(m-1)} x_i &= \sum (a_{im} - a_{i,m-1})(x_i - \alpha_i), \\ a_0^{(m)} + \sum a_i^{(m)} x_i &= -\sum a_{im}(x_i - \alpha_i), \end{aligned}$$

and it follows that

$$\sum_{k=0}^{m} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

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We shall see that the general case can be brought back to the case examined.

Theorem. *One can determine a positive parameter δ, in a manner that every closed contour C is found within the parallelohedra which are contiguous by at least one vertex, providing that the distance between any two points of the contour C does not exceed the limit δ.*

Observe, in the first place, that the distance between the points (ξ_i) and (ξ_i') belonging to the two parallelohedra which are not contiguous can not be less than a fixed limit. To demonstrate this, suppose that the point (ξ_i) belong to the parallelohedron R defined with the help of inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $R_1, R_2, \ldots, R_\sigma$ the parallelohedra which are contiguous to R and examine the set K of points belonging to the parallelohedra $R_1, R_2, \ldots, R_\sigma$.

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of parallelohedron R and designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{is}^{(h)}), \quad (h = 1, 2, \ldots, \sigma)$$

the vertices of parallelohedron $R_h \, (h = 1, 2, \ldots, \sigma)$.

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih} \alpha_{ik}^{(h)} \leq 0, \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma).$$

Designate by ρ the smallest numerical value of vertices

$$a_{0h} + \sum a_{ih} \alpha_{ik}^{(h)} \quad (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma)$$

which does not become zero. By virtue of supposition made, one will have the inequality

$$\rho + a_{0h} + \sum a_{ih} \alpha_{ik}^{(h)} \leq 0,$$

on condition that

$$a_{0h} + \sum a_{ih} \alpha_{ik}^{(h)} < 0,$$

where $(k = 1, 2, \ldots, s,\ h = 1, 2, \ldots, \sigma)$.

This established, take any point (ξ_i') which does not belong to the set K. Examine the points of a vector $[\xi_i, \xi_i']$. By putting

$$x_i = \xi_i + u(\xi_i' - \xi_i) \text{ where } 0 \leq u \leq 1,$$

let us think the parameter u of a continuous manner within the interval $0 < u < 1$. One will determine a point

$$\xi_i^{(0)} = \xi_i + u_0(\xi_i' - \xi_i) \text{ where } 0 < u_0 < 1 \tag{2}$$

which belongs to the boundary of the set K, that is to say to a face in $n - 1$ dimensions of parallelohedra $R_1, R_2, \ldots, R_\sigma$ and which also belongs to another parallelohedron R'.

Suppose that the point $(\xi_i^{(0)})$ belongs to the parallelohedron R_h. The parallelohedra R_h and R' will be contiguous by a face in $n - 1$ dimensions.

Designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{it}^{(h)}) \tag{3}$$

the vertices of parallelohedron R_h which belong to this face.

None of these vertices verifies the equation

$$a_{0h} + \sum a_{ih} x_i = 0$$

because otherwise the face examined would belong to two parallelohedron of the series $R, R_1, \ldots, R_\sigma$, which is contrary to the hypothesis.

Therefore one will have the inequalities

$$\rho + a_{0h} + \sum a_{ih} \alpha_{ik}^{(h)} \leq 0. \quad (k = 1, 2, \ldots, t)$$

The point $(\xi_i^{(0)})$ belonging to the face of R_h, which is characterised by the vertices (3), probably determined by the equations

$$\xi_i^{(0)} = \sum_{k=1}^{k=t} \vartheta_k \alpha_{ik}^{(h)} \text{ where } \sum_{k=1}^{k=t} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0.$$

$$(k = 1, 2, \ldots, t)$$

Of the previous inequalities, one draws

$$\rho + a_{0h} + \sum a_{ih}\xi_i^{(0)} \leq 0.$$

By observing that on the other hand one has

$$a_{0h} + \sum a_{ih}\xi_i \geq 0, \tag{4}$$

one finds, by (2),

$$\rho + a_{0h} + \sum a_{ih}\xi_i' < 0. \tag{5}$$

By virtue of inequalities (4) and (5), the distance $d(\xi_i, \xi_i')$ can not be smaller than a fixed limit d.

36 *Solution of the centre of the primitive parallelohedra*

This established, examine a contour C formed which the points had the mutual distance that does not surpass δ . By supposing that

$$\delta < d,$$

one will have a contour C which is situated within the contiguous parallelohedra two to two. [one to one]

Let ξ_i be any point of the contour C belonging to the parallelohedron R. Suppose that not all the points of contour C belong to R and designate by ξ_i' a point of contour C which does not belong to R. Put

$$\xi_i = \sum_{k=1}^{s} \vartheta_k \alpha_{ik} \text{ where } \sum \vartheta_k = 1 \text{ and } \vartheta_k \geq 0, \, (k = 1, 2, \ldots, s) \tag{6}$$

$$\xi_i' = \sum_{k=1}^{s} \vartheta'_k \alpha_{ik} \text{ where } \sum \vartheta'_k = 1 \tag{7}$$

As the point (ξ_i') does not belong to R, one will have among the numbers $\vartheta'_1, \vartheta'_2, \ldots, \vartheta'_s$ at least one number which will be negative. Suppose, to fix the ideas, that

$$\vartheta'_1 \geq 0, \ldots, \vartheta'_\mu \geq 0 \text{ and } \vartheta'_{\mu+1} < 0, \ldots, \vartheta'_s < 0 \tag{8}$$

One can choose the parameter δ as small that one would have the inequalities

$$|\vartheta'_k - \vartheta_k| < \epsilon, \quad (k = 1, 2, \ldots, s) \tag{9}$$

ϵ being a positive parameter also as small as one would like. By virtue of (8), it will become

$$0 \leq \vartheta_k < \epsilon, \, -\epsilon < \vartheta'_k < 0. \quad (k = \mu + 1, \ldots, s) \tag{10}$$

Observe how the largest one among the numbers ϑ_1, $\vartheta_2, \ldots, \vartheta_s$ can not be smaller than $\frac{1}{s}$ by (6) by supposing that

$$\epsilon < \frac{1}{s},$$

one wil find the required number among the number $\vartheta_1, \vartheta_2, \ldots, \vartheta_\mu$. Suppose, for fixing ideas, that

$$\vartheta_1 > \frac{1}{s}.$$

By virtue of (9), it will become

$$\vartheta'_1 > \frac{1}{s} - \epsilon \tag{11}$$

We have demonstrated that the point (ξ_i') can not belong to these parallelohedra $R_1, R_2, \ldots, R_\sigma$ which are contiguous to R. By supposing that the point (ξ_i') belong to the parallelohedron R_h, one will have an equality

$$a_{0h} + \sum a_{ih}\xi_i' < 0 \tag{12}$$

Observing that by virtue of equations (7)

$$a_{0h} + \sum a_{ih}\xi_i' = \sum_{k=1}^{s} \vartheta_k'(a_{0h} + \sum a_{ih}\alpha_{ik}),$$

one obtains, because of (12),

$$\sum_{k=1}^{s} \vartheta_k'(a_{0h} + \sum a_{ih}\alpha_{ik}) < 0$$

Of this inequality one draws, by (10) and (11),

$$\frac{1}{2}(a_{0h} \sum a_{ih}\alpha_{i1}) - \epsilon \left[a_{0h} \sum a_{ih}\alpha_{i1} + \sum_{k=\mu+1}^{s} (a_{0h} + \sum a_{ih}\alpha_{ik}) \right] < 0\,.$$

By putting

$$A = \frac{1}{2}(a_{0h} \sum a_{ih}\alpha_{i1})$$

and

$$B = a_{0h} + \sum a_{ih}\alpha_{i1} + \sum_{k=\mu+1}^{s} (a_{0h} + \sum a_{ih}\alpha_{ik}),$$

suppose that $A > 0$; the previous inequality gives $B > 0$, thus

$$\epsilon > \frac{A}{B} \tag{13}$$

Observe that the numbers A and B do not change when one replace the parallelohedron by any parallelohedron of the set (R). One concludes that the ratio $\frac{A}{B}$ which does not vanish possess a positive minimum ω.

By supposing that

$$\epsilon < \omega,$$

the inequality (13) becomes impossible and it is necessary that $A = 0$ or otherwise

$$a_{0h} + \sum a_{ih}\alpha_{i1} = 0$$

We have arrived at the following result: all the parallelohedra within which is situated the contour examined C are contiguous by the vertex (α_{i1}).

With the help of the lemma of Number 35, one will easily demonstrate the fundamental theorem stated by repeating the reasoning explained in Number 28.

37 *Fundamental properties of the generatrix function* $V(x_1, x_2, \ldots,\ x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

Theorem I. *Suppose that two vectors* $[\lambda_i]$ *and* $[\lambda_i^{(0)}]$ *characterise two parallelohedra* R *and* $R^{(0)}$ *of the set* (R). *One will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) >$$
$$V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right),$$

on condition that the point (x_i) *be interior to the parallelohedron* $R^{(0)}$.

Let$(\xi_i^{(0)})$ be any point which is interior to the parallelohedron $R^{(0)}$.

Take a point (ξ_i) which is interior to the parallelohedron R and examine a vector $[\xi_i^{(0)}, \xi_i]$ determined by the equations

$$x_i = \xi_i^{(0)} + u(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 \le u \le 1\,.$$

A group of the vector $[\xi_i^{(0)}, \xi_i]$ belong to the parallelohedron R^0. Designate,

$$\xi_i' = \xi_i^{(0)} + u_1(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 < u < 1$$

and suppose that the vector $[\xi_i^{(0)}, \xi_i']$ represents the group of the vector $[\xi_i^{(0)}, \xi_i]$ which belong to $R^{(0)}$.

The second group $[\xi_i', \xi_i]$ of the vector $[\xi_i^{(0)}, \xi_i]$ does not possess any point (ξ_i') common to the parallelohedron $R^{(0)}$. The point (ξ_i') belongs to a face $p^{(0)}(\nu)$ of the parallelohedron $R^{(0)}$. One will choose among the parallelohedra which are contiguous by the face $p^{(0)}(\nu)$ a parallelohedron R' which contains a group of the vector $[\xi_i', \xi_i]$.

Designate

$$\xi_i'' = \xi_i + u_2(\xi_i - \xi_i^{(0)}) \quad \text{where } u_1 < u_2 \leq 1$$

and suppose that the vector $[\xi_i', \xi_i'']$ represents a group of the vector $[\xi_i', \xi_i]$ which belongs to the parallelohedron R' and so on.

Let us suppose that one has determined m points of the vector $[\xi_i^{(0)}, \xi_i]$

$$\xi_i^{(k)} = \xi_i^{(0)} + u_k(\xi_i - \xi_i^{(0)})\,, \; (k = 1, 2, \ldots, m) \tag{1}$$

where

$$0 < u_1 < u_2 < \cdots < u_m < 1 \tag{2}$$

which correspond to the vectors $[\xi_i^{(0)}, \xi_i'], [\xi_i', \xi_i''], \ldots, [\xi_i^{(m)}, \xi_i]$ belonging to the parallelohedra

$$R^{(0)}, R', \ldots, R^{(m-1)}, R$$

successively contiguous.

Designate by

$$a_0^{(k)} + \sum a_i^{(k)} x_i = 0, \quad (k = 0, 1, 2, \ldots, m-1)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and $R^{(k+1)}$ which is defined within the parallelohedron $R^{(k)}$.

By virtue of the established definition in Number 32, one will have a formula

$$\begin{aligned} &V\left(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n\right) = \\ &\qquad V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right) + \\ &\qquad \textstyle\sum_{k=0}^{m-1}\left(a_0^{(k)} + \sum a_i^{(k)} x_i\right) \end{aligned} \tag{3}$$

Examine the sum

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(0)} \text{ and } a_0^{(k)} + \sum a_i^{(k)} \xi_i. \quad (k = 0, 1, 2, \ldots, m-1)$$

By virtue of the supposition made, the point $(\xi_i^{(k+1)})$ verifies the equation

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k+1)} = 0$$

As the point $(\xi^{(k)})$ belongs to the parallelohedron $R^{(k)}$, one will have an inequality

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k)} \geq 0 .$$

By virtue of (1) and (2), one obtains

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(0)} \geq 0 \text{ and } a_0^{(k)} + \sum a_i^{(k)} \xi_i \leq 0 .$$
$$(k = 0, 1, 2, \ldots, m-1)$$

As the point $(\xi_i^{(0)})$ is interior to the parallelohedron $R^{(0)}$, we will have

$$a_0^{(0)} + \sum a_i^{(0)} \xi_i^{(0)} > 0 \text{ and } a_0^{(0)} + \sum a_i^{(0)} \xi_i < 0 ,$$

It results in that

$$\sum_{k=0}^{m-1} (a_0^{(0)} + \sum a_i^{(0)} \xi_i^{(0)}) > 0 \text{ and } \sum_{k=0}^{m-1} (a_0^{(k)} + \sum a_i^{(k)} \xi_i) < 0 .$$

By substituting in the formula (3), one obtains

$$V(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}, \lambda_1, \lambda_2, \ldots, \lambda_n) > V(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$$

and

$$V(\xi_1, \xi_2, \ldots, \xi_n, \lambda_1, \lambda_2, \ldots, \lambda_n) < V(\xi_1, \xi_2, \ldots, \xi_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) .$$

Theorem II. *Suppose that the parallelohedra* $R^{(0)}, R', \ldots, R^{(n-\nu)}$ *be contiguous by a face* $p(\nu)$ *in* ν *dimensions. By designating by* $[\lambda_i^{(k)}]$, $(k = 0, 1, 2, \ldots, n-\nu)$ *the vectors which characterise these parallelohedra, one will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}),$$

on condition that the point (x_i) *be interior to a face* $p(\nu)$ *and that the vector* $[\lambda_i]$ *is not among the vectors* $[\lambda_i^{(k)}]$, $(k = 1, 2, \ldots, (n-\nu))$.

By supposing that $\lambda_i = \lambda_i^{(k)}$, one will have the equation

$$V(x_1, x_2, \ldots, x_n, \lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}) = V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}). \quad (k = 1, 2, \ldots, (n-\nu))$$

One will easily demonstrate the announced Theorem II by repeating the reasonings which have been established previously.

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The results obtained open a new way for the researches concerning the primitive parallelohedra. One can consider the set (R) of primitive parallelohedra under a new point of view, in knowing:

Each parallelohedron $R^{(0)}$ *of the set* (R) *characterised by the vector* $[\lambda_i^{(0)}]$ *presents a set of points* (x_i) *verifying* *the inequality* $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \geq V(x_i, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$, *for any vector* $[\lambda_i]$ *belonging to group* G.

We have seen in Number 32 that for the principal parallelohedron R_0 of the set (R) one has

$$V(x_i, x_2, \ldots, x_n, 0, 0, \ldots, 0) = 0.$$

It follows that the principal parallelohedron R_0 is defined by the inequality

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \geq 0$$

which holds for any vector $[\lambda_i]$ of group G.

Solution of the quadratic function $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

39

Suppose that the principal parallelohedron R_0 is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron R_k to $R_0 (k = 1, 2, \ldots, \sigma)$.

Take two parallelohedra R_k and R_0 contiguous to the parallelohedron R_0 through the faces P_k and P_h which are not parallel. Put

$$\lambda = \lambda_k + \lambda_{ih}$$

and designate by R the parallelohedron of the set (R) characterised by the vector $[\lambda_i]$.

The parallelohedron R is contiguous to the parallelohedra R_k and R_h through the faces which are congruent to the faces P_h and P_k.

One can thus form the series

$$R_0, R_k, R \text{ and } R_0, R_h, R$$

of parallelohedra which are successively contiguous.

Let us suppose that the parallelohedron R_k is determined with the help of canonical equations

$$u_k[a_{0r} + \sum a_{ir}(x_i + \lambda_{ik})] \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

The face of the parallelohedron R_k which is congruent to the face P_h will be determined by the equation

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] = 0.$$

It results in that the function $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is expressed by the sum

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ a_{0k} + \sum a_{ik}x_i + u_k\left[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})\right].$$

In the same manner, one obtains

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ a_{0h} + \sum a_{ih}x_i + u_h\left[a_{0k} + \sum a_{ik}(x_i + \lambda_{ih})\right].$$

By virtue of the fundamental theorem of Number 34, one will have an identity

$$a_{0k} + \sum a_{ik}x_i + u_k\left[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})\right] = \\ a_{0h} + \sum a_{ih}x_i + u_h\left[a_{0k} + \sum a_{ik}(x_i + \lambda_{ih})\right].$$

It follows that

$$a_{0k} + u_k(a_{0h} + \sum a_{ih}\lambda_{ik}) = a_{0h} + u_h(a_{0k} + \sum a_{ik}\lambda_{ih}) \quad (1)$$

and

$$a_{ik} + u_k a_{ih} = a_{ih} + u_h a_{ik}. \quad (i = 1, 2, \ldots, n)$$

We have supposed that the coefficients a_{ik} and a_{ih}, $(i = 1, 2, \ldots, n)$ would not be proportional , thus it is necessary that

$$u_k = 1 \text{ and } u_h = 1$$

We have arrived at the following important result:

it Any parallelohedron R characterised by a vector $[\lambda_i]$ will be determined by the canonical inequalities

$$a_{0k} + \sum a_{ik}(x_i + \lambda_i) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Observe that by virtue of (1), one will have the equation

$$\sum a_{ik}\lambda_{ih} = \sum a_{ih}\lambda_{ik}.$$

In this equation, one can attribute to the indices k and h the values $k = 1, 2, \ldots, \sigma; h = 1, 2, \ldots, \sigma$. 40

Theorem. *The vectors*

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{i\sigma}]$$

form the basis of the group G. By posing

$$\lambda_i = \sum_{k=1}^{\sigma} l_k \lambda_{ik} \tag{2}$$

where $l_1, l_2, \ldots, l_\sigma$ are arbitrary integers, one will determine each vector $[\lambda_i]$ of the group G. By indicating

$$a_i = \sum_{k=1}^{\sigma} l_k a_{ik}, \tag{3}$$

one will define the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ *by the formula*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{k=1}^{\sigma} l_k\Big(a_{0k} - \frac{1}{2}\sum_{i=1}^{n} a_{ik}\lambda_{ik} + \sum_{i=1}^{n} a_{ik}x_i\Big) + \frac{1}{2}\sum_{i=1}^{n} a_i\lambda_i \tag{4}$$

Let us suppose that the formula (4) is verified by the vectors $[\lambda_i^{(0)}]$ and $[\lambda_i']$ which are defined by the equations

$$\lambda_i^{(0)} = \sum_{k=1}^{n} l_k^{(0)}\lambda_{ik} \text{ and } \lambda_i' = \sum_{k=1}^{n} l_k'\lambda_{ik}. \ (i = 1, 2, \ldots, n) \tag{5}$$

We will see that the formula (1) will also be true for the vector $[\lambda_i]$ determined by the equations

$$\lambda_i = \lambda_i^{(0)} + \lambda_i'.$$

Let us indicate by $R, R^{(0)}$ and R' the parallelohedra characterised by the vectors $[\lambda_i], [\lambda_i^{(0)}]$ and $[\lambda_i']$.

The parallelohedron $R^{(0)}$ will be determined by the canonical inequalities

$$a_{0k} + \sum a_{ik}(x_i + \lambda_i^{(0)}) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One concludes that the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is expressed by the formula

$$\begin{aligned} V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) + \\ U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n'), \end{aligned} \tag{6}$$

Where the function $U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n')$ represents the generatrix function determined with the condition that the parallelohedron $R^{(0)}$ have been chosen for the principal parallelohedron.

By designating

$$a_i^{(0)} = \sum_{k=1}^{\sigma} l_k^{(0)} a_{ik} \text{ and } a_i' = \sum_{k=1}^{\sigma} l_k' a_{ik}, \ (i = 1, 2, \ldots, n) \qquad (7)$$

one will have, by virtue of the supposition made,

$$V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) = \\ \textstyle\sum_{k=1}^{\sigma} l_k^{(0)} \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) + \frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)},$$

$$U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n') = \\ \textstyle\sum_{k=1}^{\sigma} l_k' \left(a_{0k} + \sum a_{ik}\lambda_i^{(0)} - \frac{1}{2}\sum a_{ik}\lambda_{ik} - \sum a_{ik}x_i\right) + \frac{1}{2}\sum a_i'\lambda_i'.$$

Let us put

$$l_k = l_k^{(0)} + l_k'. \quad (k = 1, 2, \ldots, \sigma)$$

By virtue of (6) one obtains

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ \textstyle\sum l_k \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) + \\ \textstyle\frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a_i'\lambda_i' + \sum_{k=1}^{\sigma}\sum_{i=1}^{\sigma} a_{ik}l_k'\lambda_i^{(0)}. \qquad (8)$$

Let us examine the sum

$$\frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a_i'\lambda_i' + \sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik}l_k'\lambda_i^{(0)}. \qquad (9)$$

By virtue of (5), one will have

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{h=1}^{\sigma}\sum_{i=1}^{n} a_{ik}l_h^{(0)}\lambda_{ih}.$$

We have seen in Number 39 that

$$\sum_{i=1}^{n} a_{ik}\lambda_{ih} = \sum_{i=1}^{n} a_{ih}\lambda_{ik},$$

therefore

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{h=1}^{\sigma}\sum_{i=1}^{n} a_{ih} l_h^{(0)} \lambda_{ik}$$

and, because of (7), this becomes

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{i=1}^{n} a_i^{(0)} \lambda_{ik}.$$

It follows that

$$\sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik} l'_k \lambda_i^{(0)} = \sum a_i^{(0)} \lambda'_i.$$

By virtue of (7), one will also have

$$\sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik} l'_k \lambda_i^{(0)} = \sum_{i=1}^{n} a'_i \lambda_i^{(0)}.$$

One can therefore present the sum (9) under the form

$$\begin{aligned} &\frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a'_i\lambda'_i + \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik} l'_k \lambda_i^{(0)} \\ &\quad = \frac{1}{2}\left(\sum a_i^{(0)}\lambda_i^{(0)} + \sum a_i^{(0)}\lambda'_i + \sum a'_i\lambda_i^{(0)} + \sum a'_i\lambda'_i\right) \\ &\quad = \frac{1}{2}\sum \left(a_i^{(0)} + a_i^{(0)})(\lambda_i^{(0)} + \lambda'_i\right). \end{aligned}$$

As

$$a_i^{(0)} + a'_i = a_i \text{ and } \lambda_i^{(0)} + \lambda'_i = \lambda_i,$$

the formula (8) can be written

$$\begin{aligned} &V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ &\qquad \sum_{k=1}^{\sigma} l_k \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) + \frac{1}{2}\sum a_i\lambda_i. \end{aligned}$$

It is easy to verify the formula (4) in the case

$$\lambda_i = \pm\lambda_{ik}. \quad (k = 1, 2, \ldots, \sigma)$$

This results in that the formula (4) holds for any vector $[\lambda_i]$ belonging to the group G.

Theorem II. *The group G possesses a basis formed of n vector*

$$[\pi_{i1}], [\pi_{i2}], \ldots, [\pi_{in}].$$

By putting

$$\lambda_i = \sum_{k=1}^{n} l_k \pi_{ik} \tag{11}$$

where $l_1, l_2, \ldots, l_n$ are arbitrary integers, one will determine each vector $[\lambda_i]$ of the group G. By indicating

$$V(x_1, x_2, \ldots, x_n, \pi_1, \pi_2, \ldots, \pi_n) = p_{0k} + \sum_{i=1}^{n} p_i x_i,$$

$$(k = 1, 2, \ldots, n)$$

and

$$a_i = \sum_{k=1}^{n} l_k p_{ik}, \tag{12}$$

one will have the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) =$$
$$\sum_{k=1}^{n} l_k \left(p_{0k} - \frac{1}{2} \sum_{i=1}^{n} p_{ik} \pi_{ik} + \sum_{i=1}^{n} p_{ik} x_i \right) + \frac{1}{2} \sum_{i=1}^{n} a_i \lambda_i.$$

One will easily demonstrate Theorem II introduced with the help of the formula (4).

Let us notice that the sum $\sum a_i \lambda_i$ presents, by virtue of equation (11) and (12), a quadratic form of integer variables $l_1, l_2, \ldots, l_n$

$$\sum a_i \lambda_i = \sum_{k=1}^{n} \sum_{h=1}^{n} A_{kh} l_k l_h,$$

where one has put

$$A_{kh} = \frac{1}{2} \sum_{i=1}^{n} p_{ik} \pi_{ih} + \frac{1}{2} \sum_{i=1}^{n} p_{ih} \pi_{ik},$$

$$(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n)$$

We will see that the quadratic form $\sum \sum A_{kh} l_k l_h$ obtained is positive.

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Theorem I. *Any primitive parallelohedron possesses a centre.*

Designate by (ζ_i) the point satisfying the equations

$$p_{0k} - \frac{1}{2}\sum_{i=1}^{n} p_{ik}\pi_{ik} + \sum_{i=1}^{n} p_{ik}\zeta_i = 0 \quad (k = 1, 2, \ldots, n) \qquad (1)$$

I say that the point (ζ_i) represents the centre of the principal parallelohedron R_0

To demonstrate this, put

$$\lambda_{ih} = \sum_{k=1}^{n} l_k^{(h)} \pi_{ik} \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of Theorem II of Number 40, one obtains

$$\begin{aligned} V(x_1, x_2, \ldots, x_n, \lambda_{1h}, \lambda_{2h}, \ldots, \lambda_{nh}) = \\ \textstyle\sum l_k^{(h)} \left(p_{0k} - \frac{1}{2}\sum p_{ik}\pi_{ik} + \sum p_{ik}x_i\right) + \\ \textstyle\frac{1}{2}\sum_{i=1}^{n} \left(p_{i1} l_1^{(h)} + \ldots + p_{in} l_n^{(h)}\right) \lambda_{ih} \end{aligned}$$

On the other hand, by virtue of the definition established in Number 32, one has

$$V(x_1, x_2, \ldots, x_n, \lambda_{1h}, \lambda_{2h}, \ldots, \lambda_{nh}) = a_{0h} + \sum a_{ih} x_i$$

It follows that

$$a_{ih} = \sum_{k=1}^{n} l_k^{(h)} p_{ik}, \quad (h = 1, 2, \ldots, \sigma) \qquad (2)$$

and

$$\begin{aligned} a_{0h} = \textstyle\sum_{k=1}^{n} l_k^{(h)} \left(p_0 k - \frac{1}{2}\sum p_{ik}\pi_{ik}\right) + \\ \textstyle\frac{1}{2}\sum a_{ih}\lambda_{ih} \; (h = 1, 2, \ldots, \sigma) \end{aligned} \qquad (3)$$

Multiply the equation (1) by $(l^{(h)})$ and by attributing to the index k the values $1, 2, \ldots, n$, add the equations obtained, it becomes, by (2) and (3),

$$a_{0h} - \frac{1}{2}\sum a_{ih}\lambda_{ih} + \sum a_{ih}\zeta_i = 0 \, (h = 1, 2, \ldots, \sigma) \qquad (4)$$

That posed, take any one point (x_i) belonging to the parallelohedron R_0 .

For the point (ζ_i) to be the centre of the parallelohedron R_0, it is necessary and sufficient that the point (x'_i) determined by the equations

$$x'_i = 2\zeta_i - \lambda_i, \; (i = 1, 2, \ldots, n) \tag{5}$$

also belongs to the parallelohedron R_0 .

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih} x_i \geq 0. \; (h = 1, 2, \ldots, \sigma) \tag{6}$$

By noticing that by (4) and (5)

$$a_{0h} + \sum a_{ih} x'_i = -a_{0h} - \sum a_{ih}(x_i - \lambda_{ih})$$

and that the inequality

$$-a_{0h} - \sum a_{ih}(x_i - \lambda_{ih}) \geq 0$$

is found among the inequalities (6), one obtains

$$a_{0h} + \sum a_{ih} x'_i \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

It is therefore demonstrated that the point (ξ_i) represents the centre of the parallelohedron R_0.

Let us notice that the centre (ξ_i) is interior to the parallelohedron R_0.

To demonstrate this, let us suppose that a point (x_i) is interior to the parallelohedron R_0.

One will have the inequalities

$$a_{0h} + \sum a_{ih} x_i > 0. \quad (h = 1, 2, \ldots, \sigma)$$

Among these inequalities can be found the inequalities

$$-a_{0h} - \sum a_{ih}(x_i - \lambda_{ih}) > 0. \quad (h = 1, 2, \ldots, \sigma)$$

By taking the summation of these inequalities, one obtains

$$\sum a_{ih}\lambda_{ih} > 0, \quad (h = 1, 2, \ldots, \sigma)$$

and, because of the equation (4), it becomes

$$a_{0h} + \sum a_{ih}\xi_i > 0. \quad (h = 1, 2, \ldots, \sigma)$$

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Theorem II. *The quadratic form*

$$\sum_{i=1}^{n}(p_{i1}l_1 + p_{i2}l_2 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \pi_{i2}l_2 + \ldots + \pi_{in}l_n)$$

Apply Theorem I of Number 37 to the centre (ζ_i) of the principal parallelohedron R_0, one will have the inequality

$$V(\zeta_1, \zeta_2, \ldots, \zeta_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > 0, \tag{7}$$

whatever the vector $[\lambda_i]$ of the group G may be, the vector [0] being excluded.

By virtue of Theorem II of Number 40 and, [by virtue] of the equation (1), it becomes

$$p(\zeta_1, \zeta_2, \ldots, \zeta_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{2}\sum(p_{i1}l_1 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \ldots + \pi_{in}l_n)$$

and, from (7), one finds

$$\sum(p_{i1}l_1 + p_{i2}l_2 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \pi_{i2}l_2 + \ldots + \pi_{in}l_n) > 0$$

The inequality obtained holds, whatever the integer values of the variable $l_1, l_2, \ldots, l_n$ may be, the system $l_1 = 0, l_2 = 0, \ldots, l_n = 0$ being excluded.

Continuous group of the linear transformations of the primitive parallelohedra

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Applying a linear transformation of the principal primitive parallelohedron R_0 with the help of a substitution

$$x_i = \alpha_{i0} + \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \ldots, n)$$

with any real coefficients and of the determinant which does not vanish.

One obtains a new primitive parallelohedron R' which will be determined with the help of the canonical inequalities

$$a'_{0h} + \sum_{k=1}^{n} a'_{kh} x'_k \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

where one has put

$$a'_{0h} = a_{0h} + \sum_{i=1}^{n} a_{ih} \alpha_{i0}, \; a'_{kh} = \sum_{i=1}^{n} a_{ih} \alpha_{ik} \qquad (1)$$
$$(k = 1, 2, \ldots, h; h = 1, 2, \ldots, \sigma)$$

The group G' of vectors corresponding to the parallelohedron R' obtained will be determined by the equations

$$\lambda_i = \sum_{k=1}^{n} \alpha_{ik} \lambda_k, \qquad (2)$$

on condition that the vector $[\lambda_i]$ of the group G corresponds to the vector $[\lambda'_i]$ in the group G' .

Designate

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = p_0 + \sum_{i=1}^{n} p_i x_i$$

and

$$V(x'_1, x'_2, \ldots x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n) = p'_0 + \sum_{i=1}^{n} p'_i x_i$$

By virtue of the formula (4) of Number 40 and [by virtue] of the equation (1) and (2) one obtains

$$p'_0 = p_0 + \sum_{i=1}^{n} p + i\alpha_{i0},\ p'_k = \sum_{i=1}^{n} p_i \alpha_{ik} \quad (k = 1, 2, \ldots, n)$$

Of which result $[\pi_i]$ and $[\pi'_i]$ being any two corresponding vectors, one will have

$$\sum_{i=1}^{n} p_i \pi_i = \sum_{i=1}^{n} p'_i \pi'_i \qquad (3).$$

Theorem. *The quadratic form*

$$\sum_{k=1}^{n} \sum_{h=1}^{n} A_{kh} l_k l_h = \sum_{i=1}^{n} (p_{i1} l_1 + p_{i2} l_2 + \ldots + p_{in} l_n)(\pi_{i1} l_1 + \pi_{i2} l_2 + \ldots + \pi_{in} l_n)$$

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Carry out a transformation of the primitive parallelohedra of the set (R) with the help of a substitution

$$p_0 k - \frac{1}{2} \sum_{i=1}^{n} p_{ik} \pi_{ik} + \sum_{i=1}^{n} p_{ik} x_i = x'_k \quad (k = 1, 2, \ldots, n)$$

One obtains a set of the primitive parallelohedra (R').

The corresponding value of the function $V(x'_1, x'_2, \ldots, x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ for the set (R') will be expressed by the formula

$$V(x'_1, x'_2, \ldots, x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n) = \sum_{i=1}^{n} l_i x'_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} l_i l_j .$$

By virtue of the theorem Number 38, the principal parallelohedron of the set (R) will be determined by the inequalities

$$\frac{1}{2} \sum \sum A_{ij} l_i h_j + \sum_{i=1}^{n} l_i x_i \geq 0$$

which hold, whatever the integer values of $l_1, l_2, \ldots, l_n$ may be.

The various parallelohedra of the set (R') will be determined by the inequalities

$$\frac{1}{2}\sum\sum A_{ij}l_il_j + \sum l_ix_i \geq \frac{1}{2}\sum\sum A_{ij}l_i^{(0)}l_j^{(0)} + \sum l_i^{(0)}x_i \quad (4)$$

Each parallelohedron of the set (R') will be characterised by a corresponding system $(l_i^{(0)})$ of integers $l_1^{(0)}$, $l_2^{(0)}, \ldots, l_n^{(0)}$.

Observe how one could replace the base of the group G formed of n vectors by another base also formed of n vectors, these two bases will be equivalent, by virtue of Theorem III of Number 11; the corresponded positive quadratic form $\sum\sum A_{ij}l_il_j$ will be replaced by an equivalent form; the inequalities (4) define within this case the set of the parallelohedra which can be transformed as the set (R') with the help of a corresponding linear substitution on integer coefficients and of the determinant ± 1 .

The following remarkable theorem is thus demonstrated.

Theorem. *By applying the linear transformation of a primitive parallelohedron with the help of the substitutions in some real coefficients which form a group continuous for linear substitutions, one obtains a set of primitive parallelohedra which is perfectly determined by a class of equivalent positive quadratic form, on condition that one does not consider as being different the quadratic forms with proportional coefficients.*

We have seen how any positive quadratic form defines, by the help of the inequalities (4), a set of congruent parallelohedra which can be primitives or not.

Section III

Solution of the parallelohedra

with the aid of positive quadratic form

Definition of the convex polyhedron corresponding to a positive quadratic form

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Let $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ be an arbitrary positive quadratic form in n variables $x_1, x_2, \ldots, x_n$. Imagine a set R of points (α_i) satisfying the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + 2\sum_{i=1}^{n}\alpha_i x_i >\geq 0,$$

whatever may be the integer values of $x_1, x_2, \ldots, x_n$.

By virtue of the definition established, the set R enjoys the following properties:

1. The set R is in n dimensions

2. The point (0) represents the centre of the set R

3. The set R is convex.

Take we a system of arbitrary parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and examine a vector g composed of points (α_i) which are determined by the equation

$$\alpha_i = \rho\epsilon_i \quad \text{where } \rho \geq 0$$

It is easy to demonstrate that there exists an interval

$$0 \leq \rho \leq \rho_0 \quad \text{where } \rho_0 > 0$$

which correspond with the points of vector g belonging to the set R .

By posing

$$\alpha_{i0} = \rho_0\epsilon_i,$$

one obtain a vector $[\alpha_{i0}]$ the points of which belong to the set R. The point (α_{i0}) belongs to the boundary of the set R, that is to say: the point (α_{i0}) satisfies the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i0}x_i \geq 0, \tag{1}$$

whatever may be the integer values of $x_1, x_2, \ldots, x_n$ and satisfy at least one equation

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{i0}l_i = 0, \tag{2}$$

$l_1, l_2, \ldots, l_n$ being the integers which do not vanish.

Designate

$$\alpha_{i1} = -\alpha_{i0} - \sum_{j=1}^{n} a_{ij}l_j, \quad (i = 1, 2, \ldots, n) \tag{3}$$

one will have, by (2), the equation

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i1}x_i = \\ \sum\sum a_{ij}(l_i - x_i)(l_j - x_j) + 2\sum \alpha_{i0}(l_i - x_i)$$

and, by virtue of (1), one obtains

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i1}x_i \geq 0, \tag{4}$$

therefore the point (α_{i1}) also belongs to the set R .

By adding the inequalities (1) and (4), one finds, from (3),

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_j \geq 0\,.$$

The inequality obtained holds, whatever the integer values of $x_1, x_2, \ldots, x_n$; this inequality can be written

$$\sum\sum a_{ij}l_il_j \leq \sum\sum a_{ij}(l_i - 2x_i)(l_j - 2x_j)$$

One concludes that the system (l_i) is nothing but a representation of the minimum of the positive quadratic form $\sum\sum a_{ij}x_ix_j$ determined in the set composed of all the systems of integers which are contiguous to the system l_i with respect to the modulo 2.

The number of such systems is finite. Suppose that all these systems form a series

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\sigma}) \tag{5}$$

Theorem. *The set R presents a convex polyhedron determined with the aid of the inequalities*

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} \geq 0\,(k = 1, 2, \ldots, \sigma) \qquad (6)$$

By virtue of the definition established, each point (α_i) of the set R satisfies these inequalities. Suppose that a point (α_i) satisfying these inequality does not belong to the set R. One will determine in this case a positive value of the parameter ρ in the interval $0 < \rho < 1$, such that

$$\alpha_i^{(0)} = \rho\alpha_i \quad \text{where } 0 < \rho < 1, \qquad (7)$$

one obtains a point $(\alpha_i^{(0)})$ belonging to the boundary of the set R. The point $(\alpha_i^{(0)})$ will satisfy, as we have seen, an equation

$$\sum\sum a_{ij}l_i l_j + 2\sum \alpha_i^{(0)} l_i = 0 \qquad (8)$$

characterised by a system (l_i) belonging to the series (6) .

By virtue of the equation obtained, one has

$$\sum \alpha_i^{(0)} l_i < 0$$

and, by (7), it becomes

$$\sum \alpha_i l_i < 0\,.$$

By presenting the equation (8) in the form

$$\sum\sum a_{ij}l_i l_j + 2\sum \alpha_i l_i = 2(1-\rho)\sum \alpha_i l_i,$$

one will have the inequality

$$\sum\sum a_{ij}l_i l_j + 2\sum \alpha_i l_i < 0,$$

which is contrary to the hypothesis.

Independent inequalities which define the convex polyhedron corresponding to a positive quadratic form

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It may be the case that among the inequalities (6) of the previous number there are independent inequalities. Suppose, for example, that the inequality

$$\sum\sum a_ijl_il_j + 2\sum \alpha_i l_i \geq 0 \tag{1}$$

ve dependent. One will have in this case an identity

$$\begin{gathered}\sum\sum a_ijl_il_j + 2\sum \alpha_i l_i = \\ \rho_0 + \textstyle\sum_{k=1}^{\sigma} \rho_k \left(\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_ik\right)\end{gathered} \tag{2}$$

where

$$\rho_0 \geq 0, \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)\,.$$

We have seen in Number 45 that the inequality

$$\sum\sum a_{ij} x_i x_j = \sum\sum a_{ij} x_i l_j \geq 0$$

holds whatever the integer values of $x_1, x_2, \ldots, x_m$ may be.

By making in the identity (R)

$$\alpha_i = -\frac{1}{2}\sum a_{ij} l_j,$$

one obtains

$$\rho_0 + \sum \rho_k \left(\sum\sum a_{ij} l_{ik} l_{jk} - \sum\sum a_{ij} l_{ik} l_j\right) = 0$$

and consequently

$$\rho_0 = 0,\ \rho_k \left(\sum\sum a_{ij} l_{ik} l_{jk} - \sum\sum a_{ij} l_{ik} l_j\right) = 0$$

$$(k = 1, 2, \ldots, \sigma)$$

By supposing that $\rho_k \neq 0$, one will have

$$\sum\sum a_{ij} l_{ik} l_{jk} - \sum\sum a_{ij} l_{ik} l_j = 0$$

thus

$$\sum\sum a_{ij} l_i l_j = \sum\sum a_{ij} (l_i - 2l_ik)(l_j - 2l_{jk})\,.$$

By virtue of the equation obtained, the system $(l_i - 2s_{ik})$ is in the series (5) of Number 45. This is a condition necessary for the inequality (1) to be dependent.

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Theorem. *For an inequality*

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \geq 0 \tag{3}$$

to be independent, it is necessary and sufficient that the quadratic form $\sum\sum a_{ij}x_ix_j$ *does not possess as two minimum representations* (l_i) *and* $(-l_i)$ *in the set composed of all the systems of integers which are contiguous to the system* (l_i) *with regard to the modulus 2.*

We have demonstrated that the condition studied is sufficient. It remains to be demonstrated that this condition is necessary.

Let us suppose that the inequality (3) is independent. In this case

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i = 0$$

defines a face P in $n-1$ dimensions of the polyhedron R.

Let (α_i) be a point which is interior to the face P. One has the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i > 0, \tag{4}$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, the two systems (0) and (l_i) being excluded. By putting, as we have done in Number 45,

$$\alpha'_i = -\alpha_i - \sum a_{ij}l_j, \tag{5}$$

one will also have an inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha'_i x_i > 0 \tag{6}$$

which holds for any integer values of $x_1, x_2, \ldots, x_n$, the two systems (0) and (l_i) being excluded. By adding the inequality (4) and (6) one finds, by (5),

$$\sum\sum a_{ij}x_ix_j - \sum a_{ij}x_il_j > 0$$

in other words

$$\sum\sum a_{ij}l_il_j < \sum\sum a_{ij}(l_i - 2x_i)(l_j - 2x_j).$$

The inequality obtained holds for any integer values of $x_1, x_2, \ldots,$ x_n, the two systems (0) and (l_i) being excluded.

The theorem introduced is thus demonstrated.

Corollary. *The number of the independent inequalities which define the polyhedron R corresponding to a positive quadratic form can not exceed the limit* $2(2^n-1)$.

Set (R) of parallelohedra defined by a positive quadratic form.

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Theorem. *Let us suppose that the convex polyhedron R corresponding to a positive quadratic form*

$\sum\sum a_{ij}x_ix_j$ *is determined with the help of the inequalities*

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq 0.$$

By applying the translations of polyhedron R the length of the vector determined by the equations

$$\lambda_i = -\sum a_{ij}l_j,$$

$l_1, l_2, \ldots, l_n$ *being the arbitrary integers, one will make up a set (R) of congruent polyhedra which uniformly partition space in n dimensions.*

Let us indicate with R' the polyhedron which are obtained with the help of a translation of the polyhedron R the length of the vector $[\lambda_i]$. The polyhedron R' will be determined by the inequalities

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + 2\sum_{i=1}^{n}(\alpha_i + \sum_{j=1}^{n} a_{ij}l_j)x_i \geq 0.$$

This inequality can be written

$$\sum\sum a_{ij}(x_i+l_i)(x_j+l_j) + 2\sum \alpha_i(x_i+l_i) \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i.$$

One concludes that the polyhedron R' will be determined by the inequalities

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \qquad (1)$$

which hold, whatever the integer values of variables x_1, $x_2, \ldots, x_n$ may be.

One will say that the polyhedron R' congruent with the polyhedron R is characterised by the system (l_i).

Let us indicate by (R) the set of all the polyhedra congruent to polyhedron R and which are characterised by the various systems (l_i) of integers.

I argue that the set (R) uniformly fills the space in n dimensions.

Let us take an arbitrary point (α_i) in the space in n dimensions and find the polyhedron of the set (R) of which belongs the point (α_i). In this effect, determine a minimum representation (l_i) of the form

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i$$

in the set E composed of all the systems (x_i) of integer values of the variables $x_1, x_2, \ldots, x_n$.

One will have the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i$$

which holds in the set E. As a result, the point (α_i) belongs to the polyhedron of the set (R) characterised by the system (l_i).

Let us suppose that the point (α_i) belongs to the various polyhedra of the set (R): $R, R', \ldots, R^{(\mu)}$ characterised by the systems

$$(l_i), (l_{i1}), \ldots, (l_{i\mu}). \tag{2}$$

By virtue of (1), one obtains the inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i. \tag{3}$$
$$(k = 1, 2, \ldots, \mu)$$

It follows that one will have the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i > \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i,$$

for any integer values of $x_1, x_2, \ldots, x_n$, the systems (2) being excluded.

One concludes that the point (α_i) is interior to a face common to the polyhedra $R, R', \ldots, R^{(\mu)}$ and defined by the equations (3).

We have arrived at the following result: *Every positive quadratic form defines a set* (R) *of congruent parallelohedra which can be primitive or not.*

Algorithm for the search for the minimum of the form $\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i$ *in the set* E. 50

Let us suppose that one had determined the independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{jk} \geq 0 \quad (k = 1, 2, \ldots, \sigma)$$

which define the parallelohedron R corresponding to a positive quadratic form $\sum\sum a_{ij}x_ix_j$

With the help of the systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\sigma})$$

of integers, one can resolve many problems of the arithmetic theory of positive quadratic form.

We seek, for example, the minimum of the form

$$\sum\sum a_{ij}x_ix_j = 2\sum a_ix_i \qquad (1)$$

in the set E composed of all the systems (x_i) with integers, $\alpha_1, \alpha_2, \ldots,$ α_n being arbitrary parameters given.

The values of $x_1, x_2, \ldots, x_n$ which correspond to the absolute minimum of the function (1) verify the equations

$$\sum_{i=1}^{n} a_{ik}x_i + a_k = 0 \quad (k = 1, 2, \ldots, n)$$

We designate by (ξ_i) the point verifying these equations. By posing

$$\xi_i = l_i + r_i$$

we determine the integers $l_1, l_2, \ldots, l_n$ under the conditions

$$|r_i| \leq \frac{1}{2} \qquad (i = 1, 2, \ldots, n)$$

In the case $r_i = 0\,(i = 1, 2, \ldots, n)$ the system (l_i) is the one we have sought. We suppose that all the numbers $r_i\,(i = 1, 2, \ldots, n)$ do not vanish. We pose

$$\alpha_i^{(0)} = \alpha_i + \sum_{j=1}^{n} a_{ij} l_j$$

and examine the point $(\alpha_i^{(0)})$.

Let us suppose that the point (α_i) belongs to the parallelohedron of the set (R) which is characterised by the system (l_i), therefore the system represents the minimum of the form (1) .

In the case where the point $(\alpha_i^{(0)})$ does not belong to the parallelohedron R, we determine a value ρ_0 in the interval $0 < \rho_0 < 1$ of parameter ρ, in the manner such that the point $(\rho_0 \alpha_i^{(0)})$ belongs to a face of the parallelohedron R . Suppose that this face be determined by the equation

$$\sum \sum a_{ij} l_{ih} l_{jh} + 2 \sum \alpha_i l_{ih} = 0$$

One will have an equation

$$\sum \sum a_{ij} l_{ih} l_{jh} + 2\rho_0 \sum \alpha_i^0 l_{ih} = 0 \text{ where } 0 < \rho_0 < 1$$

Let

$$\alpha_i' = \alpha_i^0 + \sum_{j=1}^{n} a_{ij} l_{jh}$$

and examine anew the point (α_i') and so on. I say that one will always determine a representation of the minimum of the form (1) by repeating many times the procedure explained. To demonstrate, suppose that one had determined with the help of the algorithm shown a series of points

$$\left(\alpha_i^{(0)}\right), (\alpha_i'), \dots, \left(\alpha_i^{(k)}\right), \dots \tag{2}$$

and a series of systems

$$\left(l_i^{(0)}\right), (l_i'), \dots, \left(l_i^{(k)}\right), \dots ,$$

verifying the equations

$$\alpha_i^{(k)} = \alpha_i^{(k-1)} + \sum_{j=1}^{n} a_{ij} l_j^{(k-1)} \quad (k = 1, 2, \dots) \tag{3}$$

and the equations

$$\sum\sum a_{ij} l_i^{(k)} l_j^{(k)} + 2\sum \rho_k \alpha_i^{(k)} l_i^{(k)} = 0 \quad \text{where } 0 < \rho_k < 1 \quad (k = 0, 1, 2, \dots)$$

By virtue of these equations, one finds

$$\sum\sum a_{ij} l_i^{(k)} l_j^{(k)} + 2\sum \alpha_i^{(k)} l_i^{(k)} < 0 \quad (k = 0, 1, 2, \dots) \tag{4}$$

By designating

$$m_i^{(k)} = l_i + l_i^{(0)} + \dots + l_i^{(k-1)} \quad (k = 1, 2, \dots) \tag{5}$$

and

$$m_i^{(0)} = l_i ,$$

one obtains, from (3),

$$\alpha_i^{(k)} = \alpha_i + \sum_{j=1}^{n} a_{ij} m_j^{(k)} \quad (k = 0, 1, 2, \dots) \tag{6}$$

By substituting in the inequality (4), one gets

$$\sum\sum a_{ij}\left(l_i^{(k)} + m_i^{(k)}\right)\left(l_j^{(k)} + m_j^{(k)}\right) + 2\sum d_i \left(l_i^{(k)} + m_i^{(k)}\right) < \sum\sum a_{ij} m_i^{(k)} m_j^{(k)} + 2\sum \alpha_i m_i^{(k)}$$

This inequality, by (5), can be written

$$\sum\sum a_{ij} m_i^{k+1} m_j^{(}k+1) + 2\sum d_i m_i^{k+1} < \sum\sum a_{ij} m_i^{(k)} m_j^{(k)} + 2\sum d_i m_i^{(k)}$$

$$(k = 0, 1, 2, \ldots)$$

The number of the systems $(m_i^{(k)})$ of integers verifying these inequalities is limited. One concludes that the series of points (2) will always end by a point $(\alpha_i^{(k)})$ belonging to parallelohedron R . By virtue of the equation (6), the system $(m_i^{(k)})$ represents the minimum of the form $\sum\sum a_{ij}x_ix_j + 2\sum\sum \alpha_i x_i$ in the set E. The problem described comes down to the search for all the parallelohedra of the set (R) which are contiguous by a face in the interior of which the point $(\alpha_i^{(k)})$ is to be found. One will determine all these parallelohedra by successively determining the parallelohedra which are contiguous to R through the faces in $n-1$ dimensions and so on and so forth.

Properties of the systems of integers which characterise the faces in $n-1$ dimensions of the parallelohedron corresponding to a positive quadratic form

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Suppose that the systems

$$\pm(l_{i1}), \pm(l_{i2}), \ldots, \pm(l_{i\tau}) \tag{1}$$

characterises the faces in $n-1$ dimensions of parallelohedron R corresponding to a positive quadratic form

$$\sum\sum a_{ij}x_ix_j$$

Theorem I. *The elements $l_1k, l_2k, \ldots, l_mk$ of any system (l_{ik}) belonging to the series (1) have no common divisor.*

We have seen in Number 45 that the numbers $l_1k, l_2k, \ldots, l_nk$ verify the inequality

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_{jk} \geq 0$$

in the set E. By letting

$$l_{ik} = \delta t_i \quad \text{where } \delta \geq 1$$

and by putting $x_i = t_i$ in the previous inequality, one gets

$$\sum\sum a_{ij}t_it_j - \delta\sum\sum a_{ij}t_it_j \geq 0$$

and it is necessary that $\delta = 1$.

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Theorem II. *Suppose that n systems*

$$(p_{i1}), (p_{i2}), \dots, (p_{in}) \tag{2}$$

represent n consecutive minima

$$M_1 \leq M_2 \leq \dots M_n$$

of the positive quadratic form $\sum\sum a_{ij}x_ix_j$. All the systems (2) are in the series (1).

By virtue of the definition for the system of n consecutive minima, one will have an inequality

$$M_k = \sum\sum a_{ij}p_{ik}p_{jk} \leq \sum\sum a_{ij}x_ix_j \quad (k = 1, 2, \dots, n)$$

as long as all the numbers $x_1, x_2, \dots, x_n$ can not be presented in the form

$$M_k = \sum_{r=1}^{k-1} u_r p_i r\,,$$

the system (0) being excluded.

Suppose that the system (p_{ik}) does not belong to the series (1). In this case there exists a system (t_i) of all the numbers verifying the inequality

$$\sum\sum a_{ij}p_{ik}p_{jk} \geq \sum\sum a_{ij}(p_{ik} - 2t_i)(p_{jk} - 2t_j)$$

On letting

$$q_i = p_{ik} - t_i \tag{3}$$

one presents the previous inequality in the form

$$\sum\sum a_{ij}t_it_j + \sum\sum a_{ij}q_iq_j \tag{4}.$$

By supposing that the two systems (t_i) and (q_i) are different from the system (0), one will have, by virtue of the inequality obtained, the equation

$$t_i = \sum_{r=1}^{k-1} u_r p_i r \,, \quad q_i = \sum_{r=1}^{k-1} v_r p_i r$$

and, from (3), it follows that

$$p_{ik} = \sum_{r=1}^{k-1} (u_r + v_r) p_i r \,.$$

The equations obtained are impossible, since otherwise the determinant of n systems (2) would vanish, which is contrary to the hypothesis.

As a result, the inequality (4) does not hold at condition where either

$$t_i = 0 \quad \text{or} \quad t_i = p_{ik} \quad (i = 1, 2, \dots, n)$$

It is therefore demonstrated that the system (p_{ik}), $(k = 1, 2, \dots, n)$ belongs to the series (1).

Corollary. *All the representations for the arithmetic minimum of the positive quadratic form* $\sum\sum a_{ij} x_i x_j$ *are within the series (1).*

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Theorem III. *The numerical value of determinant of any* n *systems which belong to the series (1) is less than* $n!$

Choose any n systems in the series (1)

$$(l_{i1}), (l_{i2}), \dots, (l_{in})$$

which the determinant $\pm\omega$ does not vanish. Let us indicate

$$\alpha_{ik}^{(0)} = \frac{1}{2} \sum_{j=1}^{n} a_{ij} l_{jk} \text{ and } \alpha'_{ik} = -\frac{1}{2} \sum_{j=1}^{n} a_{ij} l_{jk} \; (k = 1, 2, \dots, n) \tag{5}$$

By virtue of the inequalities

$$\sum\sum a_{ij}x_ix_j \pm \sum\sum a_{ij}x_il_{jh} \geq 0 \quad (h = 1, 2, \dots, \tau)$$

which holds in the set E, $2n$ points (5) that belong to the parallelohedron R corresponding to the quadratic form $\sum\sum a_{ij}x_ix_j$.

Let us choose any n points among $2n$ points (5), making sure that two points corresponding to the same index k value are not among the ones chosen. One forms in this manner $2n$ systems composed of n points

$$\left(\alpha_{ih_1}^{(0)}\right), \left(\alpha_{ih_2}^{(0)}\right), \dots, \left(\alpha_{ih_\mu}^{(0)}\right), (\alpha'_{ih_{\mu+1}}), \dots, (\alpha'_{ih_n})$$

$h_1, h_2, \dots, h_n$ being any permutation of the indices $1, 2, \dots, n$ and $\mu = 0, 1, 2, \dots, n$.

We designate, to summarise,

$$\alpha_{ik}^{(h)} = \alpha_{ih_k}^{(0)},\ (k = 1, 2, \dots, \mu)\ \alpha_{ik}^{(h)} = \alpha'_{ih_k}, \qquad (6)$$
$$(k = \mu + 1, \dots, n; h = 1, 2, \dots, 2^n)$$

and examine a simplex K_h determined by the equation

$$x_i = \sum_{k=1}^{n} \vartheta_k \alpha_{ik}^{(h)} \text{ where } \sum_{k=1}^{n} \vartheta_k \leq 1 \text{ and } \vartheta_k \geq 0\ (k = 1, 2, \dots, n)$$

All the simplexes K_h, $(h = 1, 2, \dots, 2^n)$ belong to the parallelohedron R . Any point (α_i), which is interior to a simplex K_h, does not belong to any other simplex of the series formed. This results in an inequality

$$\sum_h \int_{(K_h)} dx_1dx_2\cdots dx_n < \int_{(R)} dx_1dx_2\cdots dx_n \quad (h = 1, 2, \dots, 2^n) \qquad (7)$$

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On designating by D the determinant

$$D = \begin{vmatrix} a_1 1 & a_1 2 & \cdots & a_1 n \\ \vdots & \vdots & \ddots & \vdots \\ a_n 1 & a_n 2 & \cdots & a_n n \end{vmatrix}$$

of the quadratic form $\sum\sum a_{ij}x_ix_j$, one has by virtue of (5) and (6)

$$\int_{(K_h)} dx_1 dx_2 \cdots dx_n = \frac{\omega}{n!} \cdot \frac{D}{2^n}$$

and the inequality (7) gives

$$\frac{\omega}{n!} D < \int_{(R)} dx_1 dx_2 \cdots dx_n \tag{8}$$

This established, we observe how the group G of vectors corresponding to the parallelohedron R possesses a basis formed by n vectors

$$[a_{i1}], [a_{i2}], \ldots, [a_{in}] .$$

By virtue of Theorem III of Number 11, it follows that

$$\int_{(R)} dx_1 dx_2 \cdots dx_n = D . \tag{9}$$

By substituting in the inequality (8), one would obtain

$$\omega < n!$$